Schubert Eisenstein Series

Daniel Bump and YoungJu Choie

August 11, 2011

Abstract. We define Schubert Eisenstein series as sums like usual Eisenstein series but with the summation restricted to elements of a particular Schubert cell, indexed by an element of the Weyl group. They generally not fully automorphic. We will develop some results and methods for GL_3 that may be suggestive about the general case. The six Schubert Eisenstein series are shown to have meromorphic continuation and some functional equations. The Schubert Eisenstein series $E_{s_1 s_2}$ and $E_{s_2 s_1}$ corresponding to the Weyl group elements of order three are particularly interesting: at the point where the full Eisenstein series is maximally polar, they unexpectedly become (with minor correction terms added) fully automorphic and related to each other. It is also shown for GL_3 that the Whittaker coefficients of Schubert Eisenstein series may be expressed in terms of Demazure characters.

AMS Subject Classification: 11F55.

We define *Schubert Eisenstein series* as sums like usual Eisenstein series but with the summation restricted to elements coming from a particular Schubert cell. This seems a natural notion, but little studied. The purpose of this paper is to develop some results and methods for GL_3 that suggest general lines of research for the general case.

Let G be a split reductive algebraic group over a global field F. Let $\hat{T}$ be the maximal torus of the group $\hat{G}$ with opposite root data, so that $\hat{G}(\mathbb{C})$ is the connected Langlands L-group. Let $\nu \in \hat{T}(\mathbb{C})$. Then ν parametrizes a character χ_ν of $T(\mathbb{A})/T(F)$, where $\mathbb{A}$ is the adele ring of F. Extending χ_ν to the Borel subgroup $B(\mathbb{A})$, let f_ν be an element of the corresponding induced representation, so that

$$f_\nu(bg) = (\delta^{1/2}\chi_\nu)(b)\, f(g), \qquad b \in B(\mathbb{A}). \tag{1}$$

The usual Eisenstein series is defined to be

$$E(g,\nu) = \sum_{\gamma \in B(F)\backslash G(F)} f_\nu(\gamma g) = \sum_{\gamma \in X(F)} f_\nu(\gamma g).$$

In the last expression, we are observing that the sum is actually over the integer points of $X = B\backslash G$, which is the flag variety.

The Bruhat decomposition of G gives the decomposition of the flag variety into Schubert cells

$$X = \bigcup_{w \in W} Y_w$$

where W is the Weyl group and Y_w is the image of BwB in $B\backslash G$. The closure of Y_w is the closed Schubert variety

$$X_w = \bigcup_{u \leqslant w} Y_u$$

where $\leqslant$ is the Bruhat order. It seems a natural question to consider the *Schubert Eisenstein series*

$$E_w(g,\nu) = \sum_{\gamma \in X_w(\mathbb{Z})} f_\nu(\gamma g). \tag{2}$$

This is no longer an automorphic form, but we may ask whether it has analytic continuation and at least some functional equations.

In order to see how this could be useful, let us recall the very useful Bott-Samelson varieties and their relationship with Schubert varieties. (See Bott and Samelson [1] and Demazure [8].) We will denote by α_i and s_i the simple roots and corresponding simple reflections. Let $w \in W$ and let $\mathfrak{w} = (s_{i_1}, s_{i_2}, \cdots, s_{i_k})$ be a reduced decomposition of w into a product of simple reflections: $w = s_{i_1} \cdots s_{i_k}$. Let P_j be the minimal parabolic subgroup generated by B and the one-dimensional unipotent subgroup tangent to $-\alpha_j$. We define a left action of B^k on $P_{i_1} \times \cdots P_{i_k}$ by

$$(b_1, \cdots, b_k) \cdot (p_{i_1}, \cdots, p_{i_k}) = (b_1 p_{i_1} b_2^{-1}, b_2 p_{i_2} b_3^{-1}, \cdots, b_k p_{i_k}). \tag{3}$$

The quotient $B^k\backslash(P_{i_1} \times \cdots \times P_{i_k})$ is the *Bott-Samelson variety* $Z_\mathfrak{w}$. There is a morphism $\mathrm{BS}_\mathfrak{w} : Z_\mathfrak{w} \longrightarrow X_w$ induced by the multiplication map that sends

$$(p_{i_1}, \cdots, p_{i_k}) \longmapsto p_{i_1} \cdots p_{i_k}.$$

This map is a surjective birational morphism.

Unlike the Schubert varieties, Bott-Samelson varieties are always nonsingular, so this gives a resolution of the singularities of X_w. The map $\mathrm{BS}_\mathfrak{w} : Z_\mathfrak{w} \longrightarrow X_w$ may not be an isomorphism. In special cases where it is an isomorphism, then every element of $B\backslash G$ has a unique representation as a product $\gamma_1 \cdots \gamma_k$, where γ_i is in a particular embedded copy of $B_{\mathrm{SL}_2}\backslash \mathrm{SL}_2$. Let us call this a *Bott-Samelson factorization*. (See Lemma 2 for a precise statement.) This means that we may write

$$E_{s_1 \cdots s_k}(g,\nu) = \sum_{B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} E_{s_1 \cdots s_{k-1}}(\gamma_k g, \nu), \tag{4}$$

building up the Schubert Eisenstein series by repeated SL_2 summations. Even if $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ is not an isomorphism, a modification of this method should be applicable. See Proposition 15 for an example showing how to use the geometry of the Bott-Samelson map to correct (4) when $\mathrm{BS}_{\mathfrak{w}}$ is not an isomorphism.

This method of representing the Eisenstein series $E(g, s) = E_{w_0}(g, s)$, with w_0 the long Weyl group element, is implicit in the method used by Brubaker, Bump and Friedberg [2] in order to prove that the Whittaker function of Eisenstein series on the metaplectic cover of $\mathrm{GL}_{r+1}(F)$ had a representation as a sum over a crystal basis of a representation of GL_{r+1}. The proof depends on a parametrization, described in Section 5 of the paper, of an element of $P\backslash G$, where P is a maximal parabolic subgroup, by choosing the representative factored over such a product of SL_2. Although P is a maximal parabolic subgroup, the process is an inductive one, and one could equally well avoid the induction and take the summation over $B\backslash G$. The mechanism underlying this proof therefore is the Bott-Samelson factorization.

This suggests looking more closely at the Schubert Eisenstein series E_w. Even though E_w is not automorphic, and not accessible by the usual methods of automorphic forms, one may believe that it has analytic continuation and functional equations by some subgroup. If w is the long element of a the Weyl group of the Levi subgroup M of some parabolic subgroup, then this is indeed the case, for E_w may be reduced to an Eisenstein series over M.

The first cases where w is not the long element of a Levi subgroup are $w = s_1 s_2$ and $s_2 s_1$, in the case where $G = \mathrm{GL}_3$. We will look at these Schubert Eisenstein series in detail. As it turns out, these had appeared previously, though the expression (2) had not been found. They appeared in the work of Bump and Goldfeld [5] on the Kronecker Limit formula for GL_3, in a disguised form. The published work of Bump and Goldfeld does not include details, but a portion of the calculations appeared in Bump [4], where the function Q_{ν_1,ν_2} described on page 126 may now be recognized as a Schubert Eisenstein series. The same Schubert Eisenstein series also occurs in a very disguised form in Vinogradov and Takhtajan [13].

We will take a close look at $E_{s_1 s_2}$. We have described it here by means of the definition (2) and by the recursive formula (4), but we will also see that it emerges naturally when one works out the Piatetski-Shapiro [12] Fourier-Whittaker expansion of the Eisenstein series. For a cusp form ϕ on GL_n with Whittaker function W, this Fourier expansion appears as

$$\phi(g) = \sum_{U_{\mathrm{GL}_{n-1}}(F)\backslash \mathrm{GL}_{n-1}(F)} W\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right),$$

where $U_{\mathrm{GL}_{n-1}}$ is the unipotent radical of the standard Borel subgroup of GL_{n-1}.

If ϕ is not cuspidal, then one must include other degenerate terms, and then the summation over γ may produce Schubert Eisenstein series. We will see this for GL_3.

An extremely interesting phenomenon occurs in this GL_3 case at the point where the Eisenstein series has its pole. We will choose coordinates ν_1, ν_2 for the Langlands parameters such that the poles of the Eisenstein series are on the six lines ν_1, ν_2 or $1 - \nu_1 - \nu_2$ equals 0 or $\frac{2}{3}$, and we will look at the pole at $\nu_1 = \nu_2 = 0$. In the Laurent expansion of the Eisenstein series $E(g; \nu_1, \nu_2)$ the coefficient of $\nu_1^{N_1} \nu_2^{N_2}$ is nonzero if $N_1, N_2 \geqslant -1$. If $N_1 = N_2 = -1$, the coefficent is constant. Following Bump and Goldfeld, the coefficient $\kappa(g)$ of ν_1^{-1} is then interesting.

Bump and Goldfeld [5] proved the following result. If $K/\mathbb{Q}$ is a cubic field, and $\mathfrak{a}$ is an ideal class of K one may associate with $\mathfrak{a}$ a compact torus of GL_3, and if $L_{\mathfrak{a}}$ is the period of $\kappa(g)$ over this torus, then the Taylor expansion of the L-function $L(s, \mathfrak{a})$ has the form $\rho s^{-1} + L_{\mathfrak{a}} + \cdots$. Therefore if θ is a nontrivial character of the ideal class group then $L(s, \theta) = \sum \theta(\mathfrak{a}) L_{\mathfrak{a}}$. The proof involves showing that the torus period of the Eisenstein series equals a Rankin-Selberg integral of a Hilbert modular Eisenstein series.

An analysis of this situation reveals that $\kappa(g)$ may be expressed in terms of the Schubert Eisenstein series. There are two ways to do this, giving expressions involving either $E_{s_1 s_2}$ or $E_{s_2 s_1}$ at a special value. Thus at the point where the residue is taken, the Schubert Eisenstein series (with some correction terms) is "promoted" to full GL_3 automorphicity! It is also surprising that $E_{s_1 s_2}$ and $E_{s_2 s_1}$, which are presumably unrelated in general, develop an unexpected relationship at $\nu_1 = \nu_2 = 0$.

Now let us indicate a few questions about Schubert Eisenstein series in general.

- Does the Schubert Eisenstein series always have meromorphic continuation to all values of the parameters?

- Although they will not have the full group of functional equations that the complete Eisenstein series has, they should have some functional equations.

- In Theorems 4 and 5 we will give examples of linear combinations of Schubert Eisenstein series for GL_3 that are entire, that is, have no poles in the parameters. It would be desirable to have a general theory of such linear combinations.

- In Proposition 15 we give an example of how to represent a Schubert Eisenstein series recursively in a case where the Bott-Samelson map $\mathrm{BS}_{\mathfrak{w}}$ is not an isomorphism. It would be good to work this out for more complicated examples.

- We find that for GL_3 Schubert Eisenstein series occur naturally in the context of the Piatetski-Shapiro Fourier-Whittaker expansion when one takes degenerate terms into account. It would be good to see generalizations of this phenomenon.

- We may speculate that it is possible to associate a Whittaker function with a Schubert Eisenstein series. This would be an Euler product whose p-part may be expressed in terms of Demazure characters.

Regarding the last point, we will show how to do this for the $s_1 s_2$ Schubert Eisenstein series for GL_3, but it is unclear whether one can do this in general. An affirmative answer would be quite interesting. Brubaker, Bump and Licata [3] have local results relating Iwahori Whittaker functions to Demazure characters, but we do not know how to relate those formulas to Schubert Eisenstein series.

Acknowledgement This work was supported in part by NSF grants DMS-0652817 and DMS-1001079 and by NRF-2009008-3919 and NRF-2010-0029683. We would like to thank Stanford's MRC for support, and Anthony Licata for helpful conversations about Bott-Samelson varieties.

1 Review of Eisenstein series

If G is an algebraic group defined over a field contained in a commutative ring R, we will use $G(R)$ or G_R interchangeably to denote the group of R-rational points of G.

Let F be a global field, and $\mathbb{A}$ its adele ring. Let G be a split semisimple algebraic group over F, with Borel subgroup $B = TU$, where T is its maximal split torus and U the unipotent radical. Let $W = N(T)/T$ be the Weyl group, where $N(T)$ is the normalizer of T. If v is a place of F, we will denote by $G_v = G(F_v)$, and similarly for algebraic subgroups of G. We will denote by Φ the root system of G, divided as usual into positive and negative roots Φ^+ and Φ^-. If α is a simple root, we will denote by s_α the corresponding simple reflection in W.

If v is a place of G, let K_v be a maximal compact subgroup of $G_v = G(F_v)$. We assume that $K_v = G(\mathfrak{o}_v)$ for all nonarchimedean places v. We assume that $G_v = B_v K_v$. Then $K = \prod_v K_v$ is a maximal compact subgroup of $G(\mathbb{A})$. If $w \in W$ we will choose a representative of W that is in K; by abuse of notation we will denote this representative by the same letter w.

We review the definition of the usual Eisenstein series. Let χ be a quasicharacter of $T(\mathbb{A})/T(F)$. We may extend χ_v to a quasicharacter of B_v by letting U_v be in the kernel.

Let $(\pi_v(\chi_v), V_v(\chi_v))$ be the corresponding principal series representation. Thus

$V_v(\chi_v)$ is the space of functions $f : G_v \longrightarrow \mathbb{C}$ that satisfy

$$f(bg) = \delta^{1/2}\chi(b)\, f(g)$$

for $b \in B_v = B(F_v)$, and which are K_v-finite. Here δ is the modular quasicharacter. If v is nonarchimedean the group G_v acts by right-translation:

$$\pi_v(g_v)f(x) = f(xg_v).$$

If v is archimedean, this definition is wrong since $\pi_v(g_v)f$ may not be K_v-finite, but the K_v-finite vectors are invariant under the corresponding representation of the Lie algebra $\mathfrak{g}_v$ and so at an archimedean place v, $V_v(\chi_v)$ is a $(\mathfrak{g}_v, K_v)$-module.

For simplicity we assume that $\chi = \otimes_v \chi_v$ where χ_v is unramified at every nonarchimedean place. This means that the space of K_v-fixed vectors is nonzero. The vector space $V_v(\chi_v)$ has a K_v-fixed vector $f_v^\circ = f_{\chi_v}^\circ$ that is unique up to scalar multiple. We will normalize it so that $f_v^\circ(1) = 1$.

Let $V(\chi)$ be the space of finite linear combinations of functions of the form $\prod_v f_v(g_v)$ where $f_v \in V_v(\chi_v)$ and $f_v = f_v^\circ$ for all but finitely many v. If the function f is of this form (rather than a finite linear combination of such functions) then we will write $f = \otimes_v f_v$. The space $V(\chi)$ is thus the restricted tensor product of the local modules $V_v(\chi_v)$.

Then we may consider the Eisenstein series

$$E(g, f, \chi) = \sum_{\gamma \in B_F \backslash G_F} f(\gamma g), \qquad f \in V(\chi).$$

This will be convergent for particular χ. Indeed, for every simple positive root α there is a Chevalley embedding $\iota_\alpha : \mathrm{SL}_2 \longrightarrow G$ such that $\iota_\alpha(\mathrm{SL}_2(\mathfrak{o}_v)) \subset K_v$ for v nonarchimedean, where $\mathfrak{o}_v$ is the ring of integers of F_v. Then

$$\left| \chi\left(\iota_\alpha \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right) \right| = |t|^{\nu(\alpha)}, \tag{5}$$

for some $\nu(\alpha) \in \mathbb{C}$. Indeed, since χ is trivial on $T(F)$, the left-hand side of (5) is 1 when $t \in F^\times$; then if $\mathbb{A}_1^\times$ is the group of ideles of norm 1, the left-hand side of (5) defines a homomorphism of $\mathbb{A}_1^\times / F^\times$ into the multiplicative group of positive reals. But $\mathbb{A}_1^\times / F^\times$ is compact, so the left-hand side of (5) is trivial on $\mathbb{A}_1^\times$ and thus must be a power of $|t|$. The Eisenstein series will be absolutely convergent provided every $\mathrm{re}(\nu(\alpha)) > \frac{1}{2}$. For χ not satisfying this inequality, we may make sense of the Eisenstein series by meromorphic continuation, with the exception of χ corresponding to poles of the Eisenstein series.

In order to state the functional equations of the Eisenstein series, one considers the standard intertwining integrals. If $w \in W$, define a map

$$\mathcal{M}_v(w) : V_v(\chi_v) \longrightarrow V_v(\chi_v^w),$$

where W acts on the right on quasicharacters by

$$\chi_v^w(t) = \chi_v(wtw^{-1}).$$

If $\mathrm{re}(\nu(\alpha)) > 0$, then $\mathcal{M}_v(w)$ may be defined by the integral

$$\mathcal{M}_v(w) f_v(g) = \int_{(U_v \cap w^{-1} U_v w)\backslash U_v} f_v(wug)\, du = \int_{U_v \cap w^{-1} U_v^- w} f_v(wug)\, du,$$

where U_v^- is the unipotent radical of the opposite Borel subgroup of B. It may be checked that $\mathcal{M}_v(w) V_v(\chi_v) \subseteq V_v(\chi_v^w)$, and that $\mathcal{M}_v(w)$ is an intertwining operator. The map $\mathcal{M}_v(w)$ may then be extended by meromorphic continuation to other values of χ and ν.

The formula of Gindikin and Karpelevich computes $\mathcal{M}_v(w) f_v^\circ$. First assume that v is nonarchimedean. If α is a positive root, let us denote by a_α the element

$$\iota_\alpha \begin{pmatrix} \varpi_v & \\ & \varpi_v^{-1} \end{pmatrix},$$

where ϖ_v is a generator of the maximal ideal $\mathfrak{p}_v$ of $\mathfrak{o}_v$. Let $q_v = |\mathfrak{o}_v / \mathfrak{p}_v|$. We choose the volume element dx_v on F_v so that $\mathfrak{o}_v$ has volume 1.

Proposition 1 *If v is nonarchimedean then*

$$\mathcal{M}_v(w) f^\circ_{\chi_v} = \prod_{\substack{\alpha \in \Phi^+ \\ w^{-1}(\alpha) \in \Phi^-}} \frac{1 - q_v^{-1} \chi_v(a_\alpha)}{1 - \chi_v(a_\alpha)} f^\circ_{\chi_v^w}.$$

This is called the formula of Gindikin and Karpelevich, but in this nonarchimedean case, it is due to Langlands.

Proof See Casselman [6], Theorem 3.1. $\square$

Next assume that v is archimedean. Let Γ be the usual gamma function and let

$$\Gamma_v(s) = \begin{cases} \pi^{-s/2} \Gamma(s/2) & \text{if } v \text{ is real,} \\ (2\pi)^{-s} \Gamma(s) & \text{if } v \text{ is complex.} \end{cases}$$

Since χ_v is unramified, χ_v is trivial on $T_v \cap K_v$, and it follows that

$$\chi\left(\iota_\alpha \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right) = |t|^{\nu(\alpha)}.$$

Proposition 2 *If v is archimedean then*

$$\mathcal{M}_v(w) f^\circ_{\chi_v} = \prod_{\substack{\alpha \in \Phi^+ \\ w^{-1}(\alpha) \in \Phi^-}} \frac{\Gamma_v\left(\nu(\alpha)\right)}{\Gamma_v\left(\nu(\alpha)+\frac{1}{2}\right)} f^\circ_{\chi_v^w}. \tag{6}$$

Proof This is the original formula of Gindikin and Karpelevich [9]. We are choosing the volume element on F_v to be the one that makes this formula true. □

We have choosen dx_v for every v to be the volume element that makes the formula of Gindikin and Karpelevich true. On the adele group $\mathbb{A}$ there is a natural volume element dx, which is self-dual for the Fourier transform determined by an additive character ψ on $\mathbb{A}$ that is trivial on F. Equivalently, dx is the volume element that gives $\mathbb{A}/F$ volume 1. The local and global volumes are related by the formula

$$dx = |D_F|^{-1/2} \prod_v dx_v, \tag{7}$$

where D_F is the discriminant of F.

There is also a global intertwining integral $\mathcal{M}(w) : V(\chi) \longrightarrow V(\chi^w)$, defined by

$$\mathcal{M}(w) f(g) = \int_{(U_\mathbb{A} \cap w^{-1} U_\mathbb{A} w)\backslash U_\mathbb{A}} f(wug)\, du = \int_{U_\mathbb{A} \cap w^{-1} U_\mathbb{A}^- w} f(wug)\, du$$

We are normalizing the Haar measure so that the volume of $U_\mathbb{A}$ so that $U_\mathbb{A}/U_F$ is 1, and similarly for its unipotent algebraic subgroups such as $U_\mathbb{A} \cap w^{-1} U_\mathbb{A} w$ and $U_\mathbb{A} \cap w^{-1} U_\mathbb{A}^- w$.

If α is a positive root, let

$$\zeta_v(\chi_v, \alpha) = \begin{cases} (1 - \chi_v(a_\alpha))^{-1} & \text{if } v \text{ is nonarchimedean} \\ \Gamma_v\left(\nu(\alpha)\right) & \text{if } v \text{ is archimedean.} \end{cases}$$

We will also denote

$$\zeta_v(|\cdot|\chi_v, \alpha) = \begin{cases} (1 - q_v^{-1} \chi_v(a_\alpha))^{-1} & \text{if } v \text{ is nonarchimedean,} \\ \Gamma_v\left(\nu(\alpha)+\frac{1}{2}\right) & \text{if } v \text{ is archimedean.} \end{cases}$$

Then let

$$\zeta(\chi, \alpha) = \prod_v \zeta_v(\chi_v, \alpha), \qquad \zeta(|\cdot|\chi, \alpha) = \prod_v \zeta_v(|\cdot|\chi_v, \alpha).$$

Proposition 3 *Suppose that χ is unramified at every place, and define $f^\circ_\chi \in V(\chi)$ to be $\prod_v f^\circ_{\chi_v}(g_v)$. Then*

$$\mathcal{M}(w) f^\circ_\chi = |D_F|^{l(w)/2} \prod_{\substack{\alpha \in \Phi^+ \\ w^{-1}(\alpha) \in \Phi^-}} \frac{\zeta(\chi, \alpha)}{\zeta(|\cdot|\chi, \alpha)} f^\circ_{\chi^w},$$

where $l(w)$ is the length function on the Weyl group.

Proof Because the dimension of $U \cap w^{-1}Uw$ is $l(w)$, (7) implies that, when du and du_v are the Haar measures on $U_\mathbb{A} \cap w^{-1}U^-_\mathbb{A} w$ and $U_v \cap w^{-1}U^-_v w$ with our normalizations we have

$$du = |D_F|^{l(w)/2} \prod_v du_v.$$

The statement then follows on combining (1) and (6). □

2 Induction and restriction

Mackey's theorem for finite groups and their representations may be formulated in different ways, but one statement is as follows. Let H_1 and H_2 be subgroups of G and let π_1 be representations of H_1 and H_2. We want to determine the restriction of $\mathrm{Ind}_{H_1}^G(\pi_1)$ to H_2. To answer this question we consider the double cosets $H_2\backslash G/H_1$. If w is a double coset representative, let $H_w = H_1 \cap w^{-1}H_2 w$. Then we may restrict π_1 to H_w, and conjugating by w we obtain a representation π_1^w of $wH_w w^{-1} = wH_1 w^{-1} \cap H_2$. This is a subspace of H_2, and Mackey's theorem states that

$$\mathrm{Ind}_{H_1}^G(\pi_1)|_{H_2} = \bigoplus_{w \in H_2\backslash G/H_1} \mathrm{Ind}_{wH_w w^{-1}}^{H_2}(\pi_1^w).$$

There is an analogous property of Eisenstein series. The induction and restriction functors between finite groups and subgroups will be replaced by Eisenstein series and constant term functors for Levi subgroups.

Let P and Q be parabolic subgroups of G containing B. Let $P = M_P U_P$ and $Q = M_Q U_Q$ be the Levi decompositions, with unipotent radicals U_P and U_Q contained in U. Given an automorphic form on M_Q, one may consider the corresponding Eisenstein series on G and its constant term with respect to U_P, which is an automorphic form on M_P. The problem is to describe its spectral expansion.

Using the Bruhat decomposition $G = \bigcup BwB$, representatives of double cosets $P\backslash G/Q$ may be chosen in W, and thus $P\backslash G/Q$ is in bijection with $W_P\backslash W/W_Q$, where W_P and W_Q are the Weyl groups of the Levi subgroups of P and Q. If w is such a representative, $M_Q \cap w^{-1}M_P w$ is a Levi subgroup of M_Q, so we may take the constant term along the unipotent radical of the corresponding parabolic subgroup $Q \cap w^{-1}Pw$ and obtain an automorphic form for $M_Q \cap w^{-1}M_P w$. Then conjugate this to $wM_Q w^{-1} \cap M_P$ which is an Eisenstein series on M_P. Summing over w should give an identity with the automorphic form obtained previously.

Let us prove this in the special case where $Q = B$. In this case, $M_B = T$ is the maximal torus. We will denote $M = M_P$, and $B_M = B \cap M$. We will denote by $\Phi_M \subset \Phi$ the root system of M. We will also denote by W_M the Weyl group of M, which was previously denoted W_P. If α is a root, we will denote by X_α the one parameter unipotent subgroup

$$X_\alpha(x) = \iota_\alpha \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

Lemma 1 *(i) Given a coset in W/W_M we may find a coset representative w such that if $\alpha \in \Phi_M^+$ then $w(\alpha)$ is a positive root.*

(ii) With w as in (i), if $\alpha \in \Phi_M$ then $\alpha \in \Phi_M^+$ if and only if $w(\alpha)$ is a positive root.

(iii) With w as in (i), we have

$$P \cap w^{-1}B\, w = U^w B_M, \qquad U^w = U_P \cap w^{-1}Bw.$$

Proof To prove (i), choose the representative of the coset to minimize the cardinality of $\Phi_M^+ \cap w^{-1}\Phi^-$. Such a representative w will clearly have the required property.

Then we claim that $\Phi_M^+ \cap w^{-1}\Phi^-$ is empty. Suppose not. First we will show that $\Phi_M^+ \cap w^{-1}\Phi^-$ contains a simple root. Let $\alpha \in \Phi_M^+ \cap w^{-1}\Phi^-$. Write $\alpha = \sum c_i \alpha_i$ as a linear combination of simple roots. The simple roots in Φ_M are a subset J_M of the simple roots of Φ, and if any positive root of Φ is written as a linear combination of simple roots, that root is in Φ_M if and only if only the roots in J_M appear with nonzero coefficients. Therefore all of the α_i that appear with nonzero coefficient are in Φ_M^+. If none of these is in $w^{-1}\Phi^-$ then all are in $w^{-1}\Phi^+$ so their linear combination α is in $w^{-1}\Phi^+$, which is a contradiction. Therefore there is a simple root α_i in $\Phi_M^+ \cap w^{-1}\Phi^-$.

Let us change notation now and denote α_i as just α. Now consider $\Phi_M^+ \cap (ws_\alpha)^{-1}\Phi^-$. We will show that the cardinality of this set is less than the cardinality of $\Phi_M^+ \cap w^{-1}\Phi^-$, which is a contradiction. Indeed, left multiplying by s_α shows that this set is in bijection with

$$s_\alpha \Phi_M^+ \cap w^{-1}\Phi^- = (\Phi_M^+ \cup \{-\alpha\} - \{\alpha\}) \cap w^{-1}\Phi^-$$

which is the same as $\Phi_M^+ \cap w^{-1}\Phi^-$ less one element α.

For (ii), assume $\alpha \in \Phi_M$. We know that if $\alpha \in \Phi^+$ then $w(\alpha)$ is a positive root, and we must prove the converse. Thus assume that $\alpha \in \Phi^-$. Then $-\alpha \in \Phi^+$ and so applying (i) to $-\alpha$ we have $w(-\alpha) \in \Phi^+$. This is a contradiction if $w(\alpha) \in \Phi^+$.

For (iii), suppose that α is a root such that the X_α is contained in P. Then either $X_\alpha \subseteq U_P$ or $\alpha \in \Phi_M$. Suppose that $X_\alpha \subset P \cap w^{-1}Bw$. In the first case, $X_\alpha \subset U^w$, and in the second case, using (ii), $X_\alpha \subseteq B_M$. Therefore $X_\alpha \subseteq U^w B_M$ in either case. Since $P \cap w^{-1}Bw$ is generated by unipotent subgroups X_α of this type we have $P \cap w^{-1}Bw \subseteq U^w B_M$. The other inclusion also follows from (i) or (ii). $\square$

Let Σ_M be the particular set of representatives for W/W_M given by Lemma 1. If $g \in M(\mathbb{A})$ we will denote

$$E_M(g, f, \chi) = \sum_{B_M(F)\backslash M_F} f(\gamma g),$$

which is an Eisenstein series for the Levi subgroup M.

Theorem 1 *Let $g \in M(\mathbb{A})$.*

$$\int_{U_P(F)\backslash U_P(\mathbb{A})} E(ug, f, \chi)\, du = \sum_{w\in\Sigma_M} E_M(g, \mathcal{M}(w)f, \chi^w) \tag{8}$$

Proof We may enumerate coset representatives for $B_F\backslash G_F$ as follows. Let w run through a set of coset representatives for $B_F\backslash G_F/P_F$, and for each w let γ run through a set of coset representatives for $H_F^w\backslash P_F$, where $H^w = P \cap w^{-1}Bw$. Then $w\gamma$ runs through a complete set of coset representatives for $B_F\backslash G_F$.

Using the Bruhat decomposition, we know that we may choose the representatives for w from a set of coset representatives of W/W_M, and we choose these as in Lemma 1. Therefore $H^w = U^w B_M$ where $U^w = U_P \cap w^{-1}Bw$. Then we may further analyze $\gamma \in H_F^w\backslash P_F$ as $\gamma_U\gamma_1$ where $\gamma_1 \in B_M(F)\backslash M_F$ and $\gamma_U \in U_F^w\backslash U_F$.

We may write the left-hand side in (8) as

$$\sum_{w\in\Sigma_M} \int_{U_P(F)\backslash U_P(\mathbb{A})} \sum_{\gamma_1\in B_M(F)\backslash M_F} \sum_{\gamma_U\in U_F^w\backslash U_F} f(w\gamma_U\gamma_1 ug)\, du.$$

Since M normalizes U_P, we may interchange u and γ_1 in this expression, then telescope the integration with the summation over γ_U. After this we will write γ instead of γ_1, and obtain

$$\sum_{w\in\Sigma_M} \int_{U^w(F)\backslash U_P(\mathbb{A})} \sum_{\gamma\in B_M(F)\backslash M_F} f(wu\gamma g)\, du.$$

We may write the integral as

$$\sum_{w\in\Sigma_M}\int_{U^w(F)\backslash U^w(\mathbb{A})}\int_{U^w(\mathbb{A})\backslash U_P(\mathbb{A})}\sum_{\gamma\in B_M(F)\backslash M_F} f(wu_1u\gamma g)\,du\,du_1,$$

but the integration over the compact quotient $\int_{U^w(F)\backslash U^w(\mathbb{A})}$may be discarded since $f(wu_1g)=f(wg)$ independent of $u_1\in U^w(\mathbb{A})$. Hence we obtain

$$\sum_{w\in\Sigma_M}\sum_{\gamma\in B_M(F)\backslash M_F}(\mathcal{M}(w)f)(\gamma g)\,du\,,$$

and (8) is proved. □

3 Schubert Eisenstein series

The flag variety $X=B\backslash G$ is a projective variety. We recall its decomposition into Schubert cells. We have the Bruhat decomposition $G=\bigcup BwB$, a disjoint union over $w\in W$, and let Y_w be the image of BwB in X. The Schubert cell X_w is the Zariski closure of Y_w. It equals

$$\bigcup_{\substack{u\in W\\ u\leqslant w}} Y_u,$$

where $u\leqslant w$ is the Bruhat order. Let G_w be the subset of G that is the union of BuB for $u\leqslant w$. It is not a subgroup. Let $X_w(F)$ be the set of $\gamma\in B_F\backslash G_F$ belonging to X_w. Thus $X_w(F)=B_F\backslash G_w(F)$. We may now define the *Schubert Eisenstein series*

$$E_w(g,f,\chi)=\sum_{\gamma\in X_w(F)} f(\gamma g).$$

As we explained in the introduction, the Bott-Samelson map is a useful tool for studying Schubert Eisenstein series. We recall that we defined a smooth variety $Z_{\mathfrak{w}}$ for every reduced word $\mathfrak{w}=(s_{i_1},\cdots,s_{i_k})$ representing the Weyl group element w, with a birational morphism $\mathrm{BS}_{\mathfrak{w}}:Z_{\mathfrak{w}}\longrightarrow X_w$.

Lemma 2 *If* $\mathrm{BS}_{\mathfrak{w}}$ *is an isomorphism then we may enumerate* $X_w(F)$ *as follows. Let* γ_i *run through* $B_{\mathrm{SL}_2}(F)\backslash\,\mathrm{SL}_2(F)$ *for* $i=1,\cdots,k$. *Then*

$$\iota_{\alpha_{i_1}}(\gamma_1)\cdots\iota_{\alpha_{i_k}}(\gamma_k) \tag{9}$$

runs through $X_w(F)$ *(without repetition).*

If $\mathrm{BS}_{\mathfrak{w}}$ is not an isomorphism, then every element of $X_w(F)$ can still be written as in (9), but the representation will not necessarily be unique. It will be unique if the element is in general position. We will give an example of this below in Proposition 15.

Proof If $\mathrm{BS}_{\mathfrak{w}}$ is an isomorphism, then we may choose the representatives for $Z_{\mathfrak{w}}$ as follows. First choose $p_{i_k} \in P_{i_k}$. We are allowed to choose this in the Levi subgroup $M_{i_k} \cong \mathrm{SL}_2$, and so we may choose this representative to be $\iota_{\alpha_{i_k}}(\gamma_k)$ with γ_k chosen from $B_{\mathrm{SL}_2} \backslash \mathrm{SL}_2$, where B_{SL_2} is the Borel subgroup of upper triangular matrices in SL_2. Then we may choose $p_{i_{k-1}}$ from $B \backslash P_{i_{k-1}}$, and again we may choose it from the Levi subgroup of $P_{i_{k-1}}$. Continuing this way, the statement is clear. $\square$

4 GL_3 Schubert Eisenstein series

Let

$$\zeta^*(s) = |D_F|^{\frac{s}{2}} \prod_v \zeta_v(s), \qquad \zeta_v = \begin{cases} (1-q_v^{-s})^{-1} & \text{if } v \text{ is nonarchimedean,} \\ \Gamma_v(s) & \text{if } v \text{ is archimedean} \end{cases}$$

where we recall that D_F is the discriminant of F. With this normalization of the Dedekind zeta function the functional equation is

$$\zeta^*(s) = \zeta^*(1-s).$$

For simplicity we will assume that the character χ is unramified at every place. In that case it may be described as follows. Let ν_1 and ν_2 be complex numbers. We may choose ν_1, ν_2 so that

$$(\delta^{1/2}\chi) \begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{pmatrix} = |y_1|^{2\nu_1+\nu_2} |y_2|^{\nu_2-\nu_1} |y_3|^{-\nu_1-2\nu_2}.$$

We will denote this character $\chi_{\nu_1\nu_2}$.

Since χ is unramified everywhere, we may also take $f = f^\circ$ where

$$f^\circ(g) = f^\circ_{\nu_1,\nu_2}(g) = \prod_v f^\circ_v(g_v).$$

Thus if $k \in K$

$$f^\circ_{\nu_1,\nu_2}\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} k \right) = |y_1|^{2\nu_1+\nu_2} |y_2|^{\nu_2-\nu_1} |y_3|^{-\nu_1-2\nu_2}.$$

Then we will denote
$$E(g;\nu_1,\nu_2)=E(g,f^\circ;\chi_{\nu_1,\nu_2}).$$
Due to the fact that the K-finite vectors are not invariant under right translation, we will sometimes restrict ourselves to g in the GL_3 of the finite adeles.

Denoting by α_1 and α_2 the simple positive roots and by ζ the we have, for every place and $\chi=\chi_{\nu_1,\nu_2}$

$$\zeta_v(|\cdot|\chi,\alpha_1)=\zeta_v(3\nu_1),\quad \zeta_v(|\cdot|\chi,\alpha_2)=\zeta_v(3\nu_2),$$
$$\zeta_v(|\cdot|\chi,\alpha_1+\alpha_2)=\zeta_v(3\nu_1+3\nu_2-1).$$

The product of these three factors is the local normalizing factor for the Eisenstein series at the place v. However we wish to include a power of the discriminant in the global normalizing factor, so we use $\zeta^*(s)$ which includes gamma factors and a power of the discriminant, and define

$$E^*(g;\nu_1,\nu_2)=\zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)E(g;\nu_1,\nu_2).$$

The normalized Eisenstein series E^* is analytic except at poles where ν_1,ν_2 or $1-\nu_1-\nu_2$ equals 0 or $\frac{2}{3}$. It satisfies the functional equations

$$E^*(g;\nu_1,\nu_2)=E^*(g;w(\nu_1,\nu_2))$$

Here the action of $w\in W$ on the parameters ν_1,ν_2 is as follows. The simple reflections s_1 and s_2 send (ν_1,ν_2) to $\left(\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right)$ and $\left(\nu_1+\nu_2-\frac{1}{3},\frac{2}{3}-\nu_2\right)$ respectively. We will similarly normalize the Schubert Eisenstein series and denote

$$E^*_w(g;\nu_1,\nu_2)=\zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)E_w(g;\nu_1,\nu_2).$$

If $w=1$, then

$$E^*_1(g;\nu_1,\nu_2)=\zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)f^\circ_{\nu_1,\nu_2}(g). \tag{10}$$

For particular w, we will also define E^{**}_w with only some of the normalizing zeta functions. We will omit g from the notation.

$$E^{**}_{s_1}(\nu_1,\nu_2)=\zeta^*(3\nu_1)E_{s_1}(\nu_1,\nu_2),$$
$$E^{**}_{s_2}(\nu_1,\nu_2)=\zeta^*(3\nu_2)E_{s_2}(\nu_1,\nu_2),$$
$$E^{**}_{s_1s_2}(\nu_1,\nu_2)=\zeta^*(3\nu_1)E_{s_1s_2}(\nu_1,\nu_2),$$
$$E^{**}_{s_2s_1}(\nu_1,\nu_2)=\zeta^*(3\nu_2)E_{s_2s_1}(\nu_1,\nu_2).$$

We will also consider some linear combinations denoted $\hat{E}^*_w$ or $\hat{E}^{**}_w$ that have better decay properties. These are

$$\hat{E}^*_{s_1}(\nu_1,\nu_2) = E^*_{s_1}(\nu_1,\nu_2) - E^*_1(\nu_1,\nu_2) - E^*_1\left(\frac{2}{3}-\nu_1, \nu_1+\nu_2-\frac{1}{3}\right),$$

$$\hat{E}^{**}_{s_1}(\nu_1,\nu_2) = E^{**}_{s_1}(\nu_1,\nu_2) - \zeta^*(3\nu_1) f^\circ_{\nu_1,\nu_2}(g) - \zeta^*(3\nu_1-1) f^\circ_{\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}}(g),$$

$$\hat{E}^*_{s_2}(\nu_1,\nu_2) = E^*_{s_2}(\nu_1,\nu_2) - E^*_2(\nu_1,\nu_2) - E^*_2\left(\nu_1+\nu_2-\frac{1}{3}, \frac{2}{3}-\nu_2\right),$$

$$\hat{E}^{**}_{s_2}(\nu_1,\nu_2) = E^{**}_{s_2}(\nu_1,\nu_2) - \zeta^*(3\nu_2) f^\circ_{\nu_1,\nu_2}(g) - \zeta^*(3\nu_2-1) f^\circ_{\nu_1+\nu_2-\frac{1}{3},\frac{2}{3}-\nu_2}(g),$$

$$\hat{E}^*_{s_1s_2}(\nu_1,\nu_2) = E^*_{s_1s_2}(\nu_1,\nu_2) - E^*_{s_2}(\nu_1,\nu_2) - E^*_{s_2}\left(\frac{2}{3}-\nu_1, \nu_1+\nu_2-\frac{1}{3}\right),$$

$$\hat{E}^*_{s_2s_1}(\nu_1,\nu_2) = E^*_{s_2s_1}(\nu_1,\nu_2) - E^*_{s_1}(\nu_1,\nu_2) - E^*_{s_1}\left(\nu_1+\nu_2-\frac{1}{3}, \frac{2}{3}-\nu_2\right),$$

$$\hat{E}^{**}_{s_1s_2}(\nu_1,\nu_2) = E^{**}_{s_1s_2}(\nu_1,\nu_2) - \zeta^*(3\nu_1) E_{s_2}(\nu_1,\nu_2) - \zeta^*(3\nu_1-1) E_{s_2}\left(\frac{2}{3}-\nu_1, \nu_1+\nu_2-\frac{1}{3}\right),$$

$$\hat{E}^{**}_{s_2s_1}(\nu_1,\nu_2) = E^{**}_{s_2s_1}(\nu_1,\nu_2) - \zeta^*(3\nu_2) E_{s_1}(\nu_1,\nu_2) - \zeta^*(3\nu_2-1) E_{s_1}\left(\nu_1+\nu_2-\frac{1}{3}, \frac{2}{3}-\nu_2\right).$$

Note that

$$\begin{aligned}
\hat{E}^*_{s_1}(\nu_1,\nu_2) &= \zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)\hat{E}^{**}_{s_1}(\nu_1,\nu_2),\\
\hat{E}^*_{s_2}(\nu_1,\nu_2) &= \zeta^*(3\nu_1)\zeta^*(3\nu_1+3\nu_2-1)\hat{E}^{**}_{s_2}(\nu_1,\nu_2),\\
\hat{E}^*_{s_1s_2}(\nu_1,\nu_2) &= \zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)\hat{E}^{**}_{s_1s_2}(\nu_1,\nu_2),\\
\hat{E}^*_{s_2s_1}(\nu_1,\nu_2) &= \zeta^*(3\nu_1)\zeta^*(3\nu_1+3\nu_2-1)\hat{E}^{**}_{s_2s_1}(\nu_1,\nu_2).
\end{aligned}$$

Proposition 4 *We have*

$$\int_{U_F\backslash U_{\mathbb{A}}} E(ug;\nu_1,\nu_2)\,du = \sum_{w\in W} \mathcal{M}(w) f^\circ_{\nu_1,\nu_2}(g).$$

Moreover

$$\int_{U_F\backslash U_{\mathbb{A}}} E^*(ug;\nu_1,\nu_2)\,du = \sum_{w\in W} E^*_1(g;w(\nu_1,\nu_2)). \tag{11}$$

Here E_1 is the Schubert Eisenstein series corresponding to the identity $1 \in W$. Thus $E_1 = f^\circ$ and $E^*_1 = \zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)f^\circ$.

Proof The first formula the special case of Theorem 1 where $P = B$. For the second we need to know that

$$\zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)\,\mathcal{M}(w) f^\circ_{\nu_1,\nu_2}(g) = E^*_1(g;w(\nu_1,\nu_2)). \tag{12}$$

Using the fact that $\mathcal{M}(ww') = \mathcal{M}(w)\circ\mathcal{M}(w')$ when the length $l(ww') = l(w)+l(w')$, we are reduced to the case where w is a simple reflection. For example, if $w = s_1$, Proposition 3 implies that

$$\mathcal{M}(w) f^\circ_{\nu_1,\nu_2}(g) = |D_F|^{1/2}\frac{\zeta(3\nu_1-1)}{\zeta(3\nu_1)} f^\circ_{\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}}(g).$$

Here $\zeta(s) = \prod_v \zeta_v(s)$ is the Dedekind zeta function with its gamma factors but without the power of the discriminant. So this equals

$$\frac{\zeta^*(3\nu_1-1)}{\zeta^*(3\nu_1)} f^\circ_{\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}}(g).$$

Now using the functional equation $\zeta^*(3\nu_1-1) = \zeta^*(2-3\nu_1)$, the left-hand side of (12) equals

$$\zeta^*(2-3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1) f^\circ_{\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}}(g),$$

as required. □

First we study E_{s_1}. This is essentially a GL_2 Eisenstein series. To see this, let $P = P_1$ be the parabolic with Levi factor $M_1 = \iota_{\alpha_1}(\mathrm{SL}_2)T$. Then provided $g \in M_1(\mathbb{A})$ we have

$$E_{s_1}(g;\nu_1,\nu_2) = \sum_{\gamma\in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} f^\circ_{\nu_1,\nu_2}(\iota_{\alpha_1}(\gamma)g) = E_{M_1}(g;\nu_1,\nu_2). \tag{13}$$

Proposition 5 *The normalized Schubert Eisenstein series $E^*_{s_1}$ has meromorphic continuation to all ν_1, ν_2, and satisfies*

$$E^*_{s_1}(g;\nu_1,\nu_2) = E^*_{s_1}\left(g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right). \tag{14}$$

Furthermore

$$E^{**}_{s_1}(g;\nu_1,\nu_2) = E^{**}_{s_1}\left(g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right). \tag{15}$$

We have

$$\int_{\mathbb{A}/F} E^*_{s_1}\left(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1\end{pmatrix} g;\nu_1,\nu_2\right) dx \quad = \quad E^*_1(g;\nu_1,\nu_2) + E^*_1\left(g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right). \tag{16}$$

Proof For $h \in \mathrm{GL}_2(\mathbb{A})$,

$$h \mapsto E_{M_1}\left(\begin{pmatrix} h & \\ & 1\end{pmatrix} g;\nu_1,\nu_2\right)$$

is a GL_2 Eisenstein series, and $\zeta^*(3\nu_1)$ is its normalizing factor. The analytic continuation and functional equation (15) follows from the well-known GL_2 theory. The two factors $\zeta^*(3\nu_2)$ and $\zeta^*(3\nu_1+3\nu_2-1)$ are interchanged by the transformation $(\nu_1,\nu_2) \longmapsto \left(\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right)$. Therefore the functional equation (14) follows. The GL_2 constant term is

$$\int_{\mathbb{A}/F} E^{**}_{s_1}\left(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1\end{pmatrix} g;\nu_1,\nu_2\right) dx \quad = \quad \zeta^*(3\nu_1)E_1(g;\nu_1,\nu_2) + \zeta^*(3\nu_1-1)E_1\left(g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right),$$

which is equivalent to (16). $\square$

Proposition 6 *The truncated Eisenstein series $\hat{E}^{**}_{s_1}(g;\nu_1,\nu_2)$ is entire and of rapid decay in the the α_1 direction.*

By this we mean that

$$\hat{E}^{**}_{s_1}\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} g; \nu_1, \nu_2\right)$$

is analytic for all ν_1 and ν_2, and is of faster than polynomial decay as $|y_1/y_2| \longrightarrow \infty$, uniformly if g is in a compact set.

Proof This again follows from the theory of GL_2 Eisenstein series. We have the Fourier expansion

$$E^{**}_{s_1}(g) = \sum_{\alpha \in F} \int_{\mathbb{A}/F} E^{**}_{s_1}\left(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} g\right) \psi(\alpha x)\, dx.$$

Using (16) the pieces that are subtracted to give $\hat{E}^{**}_{s_1}$ are the contribution of $\alpha = 0$. On the other hand if $\alpha \neq 0$

$$\int_{\mathbb{A}/F} E^{**}_{s_1}\left(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} g\right) \psi(\alpha x)\, dx = W\left(\begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix} g\right)$$

where

$$W(g) = \int_{\mathbb{A}/F} E^{**}_{s_1}\left(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} g\right) \psi(x)\, dx$$

is essentially a GL_2 Whittaker function. The analytic continuation of W to all ν_1, ν_2 is Théorème 1.9 of Jacquet [10], and its decay properties guarantee that

$$\hat{E}^{**}_{s_1}(g) = \sum_{\alpha \in F^\times} W\left(\begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix} g\right)$$

is entire and of rapid decay in the α_1 direction. □

Similarly

Proposition 7 *The normalized Schubert Eisenstein series $E^*_{s_2}$ has meromorphic continuation to all ν_1, ν_2, and satisfies*

$$E^*_{s_2}(g; \nu_1, \nu_2) = E^*_{s_2}\left(g; \nu_1 + \nu_2 - \frac{1}{3}, \frac{2}{3} - \nu_2\right). \tag{17}$$

Moreover $\hat{E}^{**}_{s_2}(g; \nu_1, \nu_2)$ is entire and is of rapid decay in the α_2 direction.

We turn now to the Schubert Eisenstein series $E_{s_1 s_2}$ and $E_{s_2 s_1}$. These are important examples since $s_1 s_2$ and $s_2 s_1$ are not long elements in Levi subgroups of the Weyl group, so their analytic properties do not follow from the usual theory of Eisenstein series.

Using (17) we have

$$\hat{E}^*_{s_1 s_2}(\nu_1,\nu_2) = E^*_{s_1 s_2}(\nu_1,\nu_2) - E^*_{s_2}(\nu_1,\nu_2) - E^*_{s_2}\left(\nu_2, 1-\nu_1+\nu_2\right). \tag{18}$$

Similarly

$$\hat{E}^*_{s_2 s_1}(\nu_1,\nu_2) = E^*_{s_2 s_1}(\nu_1,\nu_2) - E^*_{s_1}(\nu_1,\nu_2) - E^*_{s_1}\left(1-\nu_1+\nu_2, \nu_1\right). \tag{19}$$

Lemma 3 *Let $g \in G$. Let $f = f^\circ_{\nu_1,\nu_2}$. Then there exists a constant C depending only on g such that*

$$|f(hg)| < C|f(h)|.$$

Proof We write $h = bk$ where $b \in B(F)$ and $k \in K$. Then since $f = f^\circ$

$$|f(hg)| = |(\delta^{1/2}\chi)(b)||f(kg)| = |f(h)|\,|f(kg)|.$$

Since K is compact, $C = \max_K |f(kg)| < \infty$. □

Proposition 8 *The function*

$$\sum_{\gamma \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} \hat{E}^{**}_{s_1}(\iota_{\alpha_2}(\gamma)g;\nu_1,\nu_2) \tag{20}$$

is entire in ν_1 and ν_2.

Proof We know that $\hat{E}^{**}_{s_1}$ is entire but we need to show that the sum over γ is convergent for all ν_1 and ν_2. If $\gamma \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)$ consider

$$\begin{pmatrix} 1 & \\ & \gamma \end{pmatrix} g = \begin{pmatrix} y_1(\gamma) & * & * \\ & y_2(\gamma) & * \\ & & y_3(\gamma) \end{pmatrix} k, \qquad k \in K.$$

We will show that if $\sigma > 1$ then

$$\sum_\gamma \left| \frac{y_1(\gamma)}{y_2(\gamma)} \right|^{-2\sigma} < \infty. \tag{21}$$

Applying the Lemma to the function

$$f\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} k\right) = \left|\frac{y_1}{y_2}\right|^{-2\sigma},$$

we may assume $g = 1$ in order to prove (21). Then we note that since $\gamma \in \mathrm{SL}_2$, we have $y_1(\gamma) = 1$ and $y_2(\gamma)y_3(\gamma) = 1$. Thus $y_1(\gamma)/y_2(\gamma) = \sqrt{y_3(\gamma)/y_2(\gamma)}$, and so we must show

$$\sum_{\gamma} \left|\frac{y_2(\gamma)}{y_3(\gamma)}\right|^{\sigma} < \infty.$$

This however is a GL_2 Eisenstein series and converges if $\sigma > 1$. Now due to the rapid decay of $\hat{E}^{**}_{s_1}$ in the α_1 direction, we have

$$\hat{E}^{**}_{s_1}\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} \ll \left|\frac{y_1}{y_2}\right|^{-2\sigma}$$

as $|y_1/y_2| \longrightarrow \infty$ for any σ. Thus the estimate (21) implies the convergence of (20). □

For $w = s_1 s_2$, the Schubert variety $X_{s_1 s_2}$ coincides with the Bott-Samelson variety $Z_{(s_1,s_2)}$, since the rational map $Z_{(s_1,s_2)} \longrightarrow X_{s_1 s_2}$ is an isomorphism.

Theorem 2 *$E^*_{s_1 s_2}(g;\nu_1,\nu_2)$ has meromorphic continuation to all ν_1,ν_2. It has a functional equation*

$$E^*_{s_1 s_2}(g;\nu_1,\nu_2) = E^*_{s_1 s_2}\left(g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right).$$

Moreover $\hat{E}^{**}_{s_1 s_2}(g;\nu_1,\nu_2)$ is an entire function.

Proof When $w = s_1 s_2$ and $\mathfrak{w} = (s_1,s_2)$ the Bott-Samelson homomorphism $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ is an isomorphism and so by Lemma 2 we may write

$$E^*_{s_1 s_2}(g;\nu_1,\nu_2) = \sum_{\gamma \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} E^*_{s_1}(\iota_{\alpha_2}(\gamma)g;\nu_1,\nu_2). \tag{22}$$

Write this

$$\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)\sum_{\gamma\in B_{\mathrm{SL}_2}(F)\backslash\,\mathrm{SL}_2(F)}\hat{E}^{**}_{s_1}(\iota_{\alpha_2}(\gamma)g;\nu_1,\nu_2)$$
$$+\sum E^*_1(\iota_{\alpha_2}(\gamma)g;\nu_1,\nu_2)$$
$$+\sum E^*_1\left(\iota_{\alpha_2}(\gamma)g;\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right).$$

The meromorphic continuation of each term is known; for the first term this is by Proposition 8. Moreover, dividing by $\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)$ and rearranging gives

$$\hat{E}^{**}_{s_1s_2}(g;\nu_1,\nu_2)=\sum_{\gamma\in B_{\mathrm{SL}_2}(F)\backslash\,\mathrm{SL}_2(F)}\hat{E}^{**}_{s_1}(\iota_{\alpha_2}(\gamma)g;\nu_1,\nu_2),$$

so it follows from Proposition 8 that $\hat{E}^{**}_{s_1s_2}(g;\nu_1,\nu_2)$ is entire. □

5 Fourier-Whittaker expansion

The Fourier-Whittaker expansion of a GL_n cusp form was described by Piatetski-Shapiro [12] and is standard. We recall the proof, adapted to Eisenstein series on GL_3. We will see that the case of an Eisenstein series produces more terms beyond the Whittaker expansion that one has for a cusp form, but that these terms may be conveniently expressed in terms of Schubert Eisenstein series.

Before specializing to the Eisenstein series, let $E(g)$ denote an arbitrary automorphic form on GL_3. Making a Fourier expansion,

$$E(g)=\sum_{c,d\in F}E^c_d(g), \tag{23}$$

where

$$E^c_d(g)=\int_{(\mathbb{A}/F)^2}E\left(\begin{pmatrix}1&&x_3\\&1&x_2\\&&1\end{pmatrix}g\right)\psi(cx_3+dx_2)\,dx_2\,dx_3.$$

We recall that ψ is a nontrivial additive character on $\mathbb{A}/F$.

Proposition 9 *If* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F)$ *then*

$$E_1^0\left(\begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} g\right) = E_d^c(g). \tag{24}$$

Proof The left-hand side equals

$$\int_{(\mathbb{A}/F)^2} E\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} g\right)\psi(x_2)\,dx_2\,dx_3.$$

Making the variable change $(x_3, x_2) \mapsto (ax_3 + bx_2, cx_3 + dx_2)$ this equals

$$\int_{(\mathbb{A}/F)^2} E\left(\begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g\right)\psi(cx_3 + dx_2)\,dx_2\,dx_3,$$

and the first matrix may be discarded since E is automorphic. $\square$

It follows from (24) that $E_1^0\left(\begin{pmatrix} 1 & x_1 & \\ & 1 & \\ & & 1 \end{pmatrix} g\right)$ is periodic and may be regarded as a function of $x_1 \in \mathbb{A}/F$. Therefore by Fourier inversion

$$E_1^0(g) = \sum_{n \in F} E_{n,1}(g) \tag{25}$$

where

$$E_{n,1}(g) = \int_{\mathbb{A}/F} E_1^0\left(\begin{pmatrix} 1 & x_1 & \\ & 1 & \\ & & 1 \end{pmatrix} g\right)\psi(nx_1)\,dx_1.$$

More generally if $n_1, n_2 \in F$ define

$$E_{n_1,n_2}(g) = \int_{(\mathbb{A}/F)^3} E\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g\right)\psi(n_1x_1 + n_2x_2)\,dx_1\,dx_2\,dx_3.$$

If $n = 1$, then $W(g) = E_{1,1}(g)$ is the standard *Whittaker function*

$$W(g) = \int_{(\mathbb{A}/F)^3} E\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g\right)\psi(x_1 + x_2)\,dx_1\,dx_2\,dx_3$$

Proposition 10 *If* $0 \neq n \in F$ *then*

$$E_{n,1}(g) = W\left(\begin{pmatrix} n & & \\ & 1 & \\ & & 1 \end{pmatrix} g\right).$$

Proof This is similar to the proof of Proposition 9. □

Theorem 3 *We have*

$$E(g) = E_0^0(g) + \sum_{\gamma \in U_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} E_{0,1}(\iota_{\alpha_1}(\gamma) g) + \sum_{\gamma \in U_{\mathrm{GL}_2}(F)\backslash \mathrm{GL}_2(F)} W\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \tag{26}$$

Here $U_{\mathrm{GL}_2} = U_{\mathrm{SL}_2}$ is the one parameter subgroup $\iota_{\alpha_1}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$.

Proof Combining (23) and (24)

$$E(g) = E_0^0(g) + \sum_{\gamma \in U_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} E_1^0(\iota_{\alpha_1}(\gamma) g).$$

Indeed, given $(c, d) \neq (0, 0)$ in F^2, we may find $\gamma \in \mathrm{SL}_2(F)$ with bottom row (c, d), and then $E_1^0(\iota_{\alpha_1}(\gamma) g) = E_d^c(g)$. But γ is determined up to left multiplication by an element of $U_{\mathrm{SL}_2}(F)$.

Now we use (25) and separate out the cases where $n = 0$ and $n \neq 0$. The contribution when $n = 0$ accounts for the second term in (26). For the remaining terms by Proposition 10 we obtain

$$\sum_{\gamma \in U_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} \sum_{n \in F^\times} W\left(\begin{pmatrix} n & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right),$$

and combining the two matrices this is equivalent to summing over $U_{\mathrm{GL}_2}(F)\backslash \mathrm{GL}_2(F)$. □

Now let us consider the case where $E(g) = E^*(g; \nu_1, \nu_2)$.

Proposition 11 *We have*

$$\int_{(\mathbb{A}/F)^2} E^*\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g; \nu_1, \nu_2\right) dx_2\, dx_3 \;=\; E^*_{s_1}(g;\nu_1,\nu_2) + E^*_{s_1}(g; 1-\nu_1-\nu_2, \nu_1) + E^*_{s_1}(g;\nu_2, 1-\nu_1-\nu_2).$$

This is $E_0^0(g)$ when $E(g) = E^*(g;\nu_1,\nu_2)$.

Proof This is a special case of Theorem 1. The three double coset representatives in Σ_M are

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \quad \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}.$$

Using (13) the corresponding GL_2 Eisenstein series may be written as

$$E^*_{s_1}(g;\nu_1,\nu_2), \qquad E^*_{s_1}\left(g; \nu_1+\nu_2-\frac{1}{3}, \frac{2}{3}-\nu_2\right), \qquad E^*_{s_1}\left(g; \frac{2}{3}-\nu_1, \nu_1+\nu_2-\frac{1}{3}\right),$$

and using the functional equations these are the three terms in the statement. □

Proposition 12 *We have*

$$\int_{(\mathbb{A}/F)^3} E^*\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & \\ & & 1 \end{pmatrix} g; \nu_1, \nu_2\right) dx_1\, dx_3 \;=\; E^*_{s_2}(g;\nu_1,\nu_2) + E^*_{s_2}(g; 1-\nu_1-\nu_2, \nu_1) + E^*_{s_2}(g;\nu_2, 1-\nu_1-\nu_2).$$

Proof This is similar to Proposition 11 except that we use the other maximal parabolic subgroup. □

Proposition 13 *If $E(g) = E^*(g;\nu_1,\nu_2)$ then*

$$\sum_{\gamma \in U_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} E_{0,1}(\iota_{\alpha_1}(\gamma) g) \;=\; E^*_{s_2 s_1}(g;\nu_1,\nu_2) + E^*_{s_2 s_1}(g; 1-\nu_1-\nu_2, \nu_1) + E^*_{s_2 s_1}(g;\nu_2, 1-\nu_1-\nu_2)$$
$$-2(E^*_{s_1}(g;\nu_1,\nu_2) + E^*_{s_1}(g; 1-\nu_1-\nu_2, \nu_1) + E^*_{s_1}(g;\nu_2, 1-\nu_1-\nu_2))$$

Proof We may write the left-hand side as

$$\sum_{\gamma\in B_{\mathrm{SL}_2(F)}\backslash \mathrm{SL}_2(F)} \sum_{n\in F^*} E_{0,1}\left(\begin{pmatrix} n^{-1} & & \\ & n & \\ & & 1\end{pmatrix} \iota_{\alpha_1}(\gamma) g\right).$$

Similarly to Proposition 10 we have

$$E_{0,1}\left(\begin{pmatrix} n^{-1} & & \\ & n & \\ & & 1\end{pmatrix} g\right) = E_{0,n}(g)$$

so the left-hand side equals

$$\sum_{\gamma\in B_{\mathrm{SL}_2(F)}\backslash \mathrm{SL}_2(F)} \sum_{n\in F^\times} E_{0,n}(\iota_{\alpha_1}(\gamma)g).$$

We will show that

$$\sum_{\gamma\in B_{\mathrm{SL}_2(F)}\backslash \mathrm{SL}_2(F)} \sum_{n\in F} E_{0,n}(\iota_{\alpha_1}(\gamma)g) = \\ E^*_{s_2s_1}(g;\nu_1,\nu_2) + E^*_{s_2s_1}(g;1-\nu_1-\nu_2,\nu_1) + E^*_{s_2s_1}(g;\nu_2,1-\nu_1-\nu_2) \tag{27}$$

and that

$$\sum_{\gamma\in B_{\mathrm{SL}_2(F)}\backslash \mathrm{SL}_2(F)} E_{0,0}(\iota_{\alpha_1}(\gamma)g) = \sum_{w\in W} E^*_{s_1}(g;w(\nu_1,\nu_2)). \tag{28}$$

Combining these two identities gives the statement.

Observe that

$$\sum_{n\in F} E_{0,n}(g) = \sum_{n\in F} \int_{(\mathbb{A}/F)^3} E\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1\end{pmatrix} g\right) \psi(nx_2)\, dx_1\, dx_2\, dx_3 = \\ \int_{(\mathbb{A}/F)^3} E\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & \\ & & 1\end{pmatrix} g\right) dx_1\, dx_3,$$

which is evaluated in Proposition 12. Thus (27) is the sum of three terms, a typical one being

$$\sum_{\gamma\in B_{\mathrm{SL}_2(F)}\backslash \mathrm{SL}_2(F)} E^*_{s_2}(\iota_{\alpha_1}(\gamma)g;\nu_1,\nu_2).$$

This is $E^*_{s_2s_1}(g;\nu_1,\nu_2)$, similarly to (22), whence (27). Also note that $E_{0,0}(g)$ is evaluated above in (11), and summing over $\iota_{\alpha_1}(\gamma)$ gives

$$\sum_{w\in W} E^*_{s_1}(g; w(\nu_1,\nu_2)).$$

We note that this may be written as

$$2(E^*_{s_1}(g;\nu_1,\nu_2) + E^*_{s_1}(g; 1-\nu_1-\nu_2,\nu_1) + E^*_{s_1}(g;\nu_2, 1-\nu_1-\nu_2))$$

because of the functional equation (14). $\square$

Let

$$H(g;\nu_1,\nu_2) = \sum_{\gamma\in U_{\mathrm{GL}_2}(F)\backslash \mathrm{GL}_2(F)} W\left(\begin{pmatrix} \gamma & \\ & 1\end{pmatrix} g\right), \tag{29}$$

where

$$W(g) = \int_{(\mathbb{A}/F)^3} E^*\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1\end{pmatrix} g;\nu_1,\nu_2\right)\psi(x_1+x_2)\,dx_1\,dx_2\,dx_3.$$

Theorem 4 *The function $H(g;\nu_1,\nu_2)$ is entire as a function of ν_1 and ν_2. We have*

$$\begin{aligned} E^*(g;\nu_1,\nu_2) &= \\ H(g;\nu_1\nu_2) + & \\ E^*_{s_2s_1}(g;\nu_1,\nu_2) + E^*_{s_2s_1}(g;1-\nu_1-\nu_2,\nu_1) + E^*_{s_2s_1}(g;\nu_2,1-\nu_1-\nu_2) & \\ -E^*_{s_1}(g;\nu_1,\nu_2) - E^*_{s_1}(g;1-\nu_1-\nu_2,\nu_1) - E^*_{s_1}(g;\nu_2,1-\nu_1-\nu_2) &= \\ \hat{E}^*_{s_2s_1}(g;\nu_1,\nu_2) + \hat{E}^*_{s_2s_1}(g;1-\nu_1-\nu_2,\nu_1) + \hat{E}^*_{s_2s_1}(g;\nu_2,1-\nu_1-\nu_2) & \\ +E^*_{s_1}(g;\nu_1,\nu_2) + E^*_{s_1}(g;1-\nu_1-\nu_2,\nu_1) + E^*_{s_1}(g;\nu_2,1-\nu_1-\nu_2) & \end{aligned}$$

Proof We have

$$W(g) = \prod_v W_v(g_v)$$

where the Jacquet-Whittaker function W_v has analytic continuation for every place v by Jacquet [10], Corollaire 3.5, and the convergence of the sum in (29) follows from the decay properties of the Whittaker function (Proposition 2.2 in Jacquet, Piatetski-Shapiro and Shalika [11]. Therefore H is entire.

We note that $H(g;\nu_1,\nu_2)$ is one of the three terms in (26). The remaining terms are evaluated in Proposition 11 and Proposition 13. Combining these gives first expression. The second expression follows by using the definition of $\hat{E}^*_{s_2s_1}$. $\square$

Similarly, one may prove that if

$$H'(g;\nu_1,\nu_2) = \sum_{\gamma\in U_{\mathrm{GL}_2}(F)\backslash\,\mathrm{GL}_2(F)} W\left(\begin{pmatrix} 1 & \\ & \gamma \end{pmatrix} g\right)$$

then the following is true.

Theorem 5 *The function $H'(g;\nu_1,\nu_2)$ is entire as a function of ν_1 and ν_2. We have*

$$\begin{aligned} E^*(g;\nu_1,\nu_2) &= \\ H'(g;\nu_1\nu_2)+ & \\ E^*_{s_1s_2}(g;\nu_1,\nu_2)+E^*_{s_1s_2}(g;1-\nu_1-\nu_2,\nu_1)+E^*_{s_1s_2}(g;\nu_2,1-\nu_1-\nu_2) & \\ -E^*_{s_2}(g;\nu_1,\nu_2)-E^*_{s_2}(g;1-\nu_1-\nu_2,\nu_1)-E^*_{s_2}(g;\nu_2,1-\nu_1-\nu_2) &= \\ \hat{E}^*_{s_1s_2}(g;\nu_1,\nu_2)+\hat{E}^*_{s_1s_2}(g;1-\nu_1-\nu_2,\nu_1)+\hat{E}^*_{s_1s_2}(g;\nu_2,1-\nu_1-\nu_2) & \\ +E^*_{s_2}(g;\nu_1,\nu_2)+E^*_{s_2}(g;1-\nu_1-\nu_2,\nu_1)+E^*_{s_2}(g;\nu_2,1-\nu_1-\nu_2) & \end{aligned}$$

6 Kronecker Limit Formula

The poles of the Eisenstein series are on the six lines where ν_1, ν_2 or $1-\nu_1-\nu_2$ equals 0 or $\frac{2}{3}$. We will consider the Taylor expansions of E_w for various w at $\nu_1=\nu_2=0$. In particular, the coefficient of ν_1^{-1} is interesting. Ifϕ is a function of g,ν_1,ν_2 let $\mathfrak{R}\phi$ be the coefficient of ν_1^{-1} in the Taylor expansion of ϕ at $\nu_1=\nu_2=0$. Let

$$\kappa(g)=\mathfrak{R}E(g;\nu_1,\nu_2).$$

Bump and Goldfeld [5] proved the following result. If $K/\mathbb{Q}$ is a cubic field, and $\mathfrak{a}$ is an ideal class of K one may associate with $\mathfrak{a}$ a compact torus of GL_3, and if $L_{\mathfrak{a}}$ is the period of $\kappa(g)$ over this torus, then the Taylor expansion of the L-function $L(s,\mathfrak{a})$ has the form $\rho s^{-1}+L_{\mathfrak{a}}+\cdots$. Therefore if θ is a character of the ideal class group then $L(s,\theta)=\sum\theta(\mathfrak{a})L_{\mathfrak{a}}$. The proof involves showing that the torus period of the Eisenstein series equals a Rankin-Selberg integral of a Hilbert modular Eisenstein series.

An analysis of this situation reveals that κ may be expressed in terms of the Schubert Eisenstein series. There are two ways to do this, giving expressions involving either $E_{s_1s_2}$ or $E_{s_2s_1}$ at a special value. Thus at the point where the residue is taken, the Schubert Eisenstein series (with some correction terms) is “promoted” to full GL_3 automorphicity!

Let us write

$$\zeta^*(s) = \frac{\rho}{s} + \delta + O(s).$$

Then

$$E^{**}_{s_1}(g;\nu_1,\nu_2) = \frac{\rho}{3\nu_1} + \phi_{s_1}(g;\nu_2) + O(\nu_1)$$

where ϕ_{s_1} satisfies

$$\phi_{s_1}\left(i_{\alpha_1}(\gamma)g;\nu_2\right) = \phi_{s_1}(g;\nu_2),$$

since E_{s_1} has the same automorphicity. Similarly

$$E^{**}_{s_2}(g;\nu_1,\nu_2) = \frac{\rho}{3\nu_2} + \phi_{s_2}(g;\nu_1) + O(\nu_2).$$

We will write

$$\phi_{s_1}(g) = \phi_{s_1}(g;0), \qquad \phi_{s_2}(g) = \phi_{s_2}(g;0).$$

The automorphic forms ϕ_{s_1} and ϕ_{s_2} are essentially GL_2 automorphic forms, similar to the function $\log|\eta(z)|$ that appears in the classical Kronecker Limit Formula.

Let

$$c_0 = \frac{\rho}{3}\left[\delta\zeta^*(-1) + \rho(\zeta^*)'(-1)\right], \qquad c_0' = \frac{\rho}{3}\left[\zeta^*(3)\zeta^*(-1) + \rho\frac{d}{ds}(\zeta^*)'(-1)\right].$$

These are absolute constants depending only on the field.

Theorem 6 *We have*

$$\kappa(g) = \frac{\rho}{3}\zeta^*(2)\left[\hat{E}^{**}_{s_2s_1}(g;0,0) + E^{**}_{s_1}(g;1,0)\right] + c_0.$$

Furthermore

$$\kappa(g) = \frac{\rho}{3}\zeta^*(2)\left[\hat{E}^{**}_{s_1s_2}(g;1,0) + \phi_{s_2}(g)\right] + c_0'.$$

Proof The points $(\nu_1,\nu_2) = (0,0)$ and $(1,0)$ are related by a functional equation of the total Eisenstein series $E(g;\nu_1,\nu_2)$, but not of the Schubert Eisenstein series. We could alternatively take the Taylor coefficient of ν_2^{-1} and obtain a similar pair of identities.

By Theorem 4 we have

$$\kappa(g) = \sum_{i=1}^{6} \Re X_i$$

where X_i runs through the following six terms.

X_i	long form	$\Re X_i$
$\hat{E}^*_{s_2s_1}(g;\nu_1,\nu_2)$	$\zeta^*(3\nu_1)\zeta^*(3\nu_1+3\nu_2-1)$ $\hat{E}^{**}_{s_2s_1}(g;\nu_1,\nu_2)$	$\frac{\rho}{3}\zeta^*(-1)\hat{E}^{**}_{s_2s_1}(g;0,0)$
$\hat{E}^*_{s_2s_1}(g;1-\nu_1-\nu_2,\nu_1)$	$\zeta^*(3-3\nu_1-3\nu_2)\zeta^*(2-3\nu_2)$ $\hat{E}^{**}_{s_2s_1}(g;1-\nu_1-\nu_2,\nu_1)$	0
$\hat{E}^*_{s_2s_1}(g;\nu_2,1-\nu_1-\nu_2)$	$\zeta^*(3\nu_2)\zeta^*(2-3\nu_1)$ $\hat{E}^{**}_{s_2s_1}(g;\nu_2,1-\nu_1-\nu_2)$	0
$E^*_{s_1}(g;\nu_1,\nu_2)$	$\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)$ $E^{**}_{s_1}(g;\nu_1,\nu_2)$	c_0
$E^*_{s_1}(g;1-\nu_1-\nu_2,\nu_1)$	$\zeta^*(3\nu_1)\zeta^*(2-3\nu_2)$ $E^{**}_{s_1}(g;1-\nu_1-\nu_2,\nu_1)$	$\frac{\rho}{3}\zeta^*(-1)E^{**}_{s_1}(g;1,0)$.
$E^*_{s_1}(g;\nu_2,1-\nu_1-\nu_2)$	$\zeta^*(3-3\nu_1-3\nu_2)\zeta^*(2-3\nu_1)$ $E^{**}_{s_1}(g;\nu_2,1-\nu_1-\nu_2)$	0

Alternatively, by Theorem 5 we may use the following six terms:

X_i	long form	$\Re X_i$
$\hat{E}^*_{s_1s_2}(\nu_1,\nu_2)$	$\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)$ $\hat{E}^{**}_{s_1s_2}(g;\nu_1,\nu_2)$	0
$\hat{E}^*_{s_1s_2}(1-\nu_1-\nu_2,\nu_1)$	$\zeta^*(3\nu_1)\zeta^*(2-3\nu_2)$ $\hat{E}^{**}_{s_1s_2}(1-\nu_1-\nu_2,\nu_1)$	$\frac{\rho}{3}\zeta^*(-1)\hat{E}^{**}_{s_1s_2}(1,0)$
$\hat{E}^*_{s_1s_2}(g;\nu_2,1-\nu_1-\nu_2)$	$\zeta^*(3-3\nu_1-3\nu_2)\zeta^*(2-3\nu_1)$ $\hat{E}^{**}_{s_1s_2}(\nu_2,1-\nu_1-\nu_2)$	0
$E^*_{s_2}(g;\nu_1,\nu_2)$	$\zeta^*(3\nu_1)\zeta^*(3\nu_1+3\nu_2-1)$ $E^{**}_{s_2}(g;\nu_1,\nu_2)$	$\frac{\rho}{3}\zeta^*(-1)\phi_{s_2}(g)$ $+\frac{\rho^2}{3}(\zeta^*)'(-1)$
$E^*_{s_2}(g;1-\nu_1-\nu_2,\nu_1)$	$\zeta^*(3-3\nu_1-3\nu_2)\zeta^*(2-3\nu_2)$ $E^{**}_{s_2}(g;1-\nu_1-\nu_2,\nu_1)$	$\zeta^*(3)\zeta^*(-1)\frac{\rho}{3}$.
$E^*_{s_2}(g;\nu_2,1-\nu_1-\nu_2)$	$\zeta(3\nu_2)\zeta^*(2-3\nu_1)$ $E^{**}_{s_2}(g;\nu_2,1-\nu_1-\nu_2)$	0

□

7 When $\mathrm{BS}_{\mathfrak{w}}$ is not an isomorphism

Let w_0 be the long Weyl group element. The Schubert Eisenstein series E_{w_0} is then just the full Eisenstein series, which is well understood. Nevertheless, we may try to understand it as a Schubert Eisenstein series.

For GL_3, there are two reduced words $\mathfrak{w} = (s_1, s_2, s_1)$ or (s_2, s_1, s_2) representing w_0. If $\mathfrak{w}$ is either of these, the Bott-Samelson homomorphism $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_{w_0} = X$ is not an isomorphism. However, since it is birational, it is a local isomorphism on the complement of a closed subvariety, which may be described as follows. The space X may be identified with the space of full flags in a 3-dimensional vector subspace V. Let $V_0 \subset V_1 \subset V_2 \subset V_3$ be the standard flag, where V_i is the span of $e_1, \cdots, e_i$, in terms of the standard basis vectors e_i of V.

Proposition 14 *With $\mathfrak{w} = (s_1, s_2, s_1)$, $Z_{\mathfrak{w}}$ may be identified with the space of flags $V_0 \subset U_1 \subset U_2 \subset V_3$ with an auxiliar piece of data, namely a one-dimensional vector space W_1 such that $W_1 \subset V_2 \cap U_2$.*

Proof To see this, consider the sequence of flags:

$$\begin{array}{ccccccc}
V_3 & & V_3 & & V_3 & & V_3\\
| & & | & & | & & |\\
V_2 & & V_2 & & U_2 & & U_2\\
| & \xleftarrow{\theta_1} & | & \xleftarrow{\theta_2} & | & \xleftarrow{\theta_3} & |\\
V_1 & & W_1 & & W_1 & & U_1\\
| & & | & & | & & |\\
V_0 & & V_0 & & V_0 & & V_0
\end{array}$$

We select elements θ_1, θ_2 and θ_3 of GL_3 such that θ_1 takes the second flag to the first, θ_2 takes the third to the second, and θ_3 takes the last to the third. Then θ_1 is in the parabolic subgroup P_1 that fixes the partial flag $V_0 \subset V_2 \subset V_3$, θ_2 stabilizes the partial flag $V_0 \subset W_1 \subset V_3$ and θ_3 fixes the partial flag $V_0 \subset U_2 \subset V_3$. This means that $\theta_1\theta_2^{-1}\theta_1^{-1}$ is in the parabolic subgroup P_2 that fixes the partial flag $V_0 \subset V_1 \subset V_3$ and similarly $\theta_1\theta_2\theta_3^{-1}\theta_2^{-1}\theta_1^{-1}$ is in P_1. Let us consider $(p_1, p_2, p_3) = (\theta_1^{-1}, \theta_1\theta_2^{-1}\theta_1^{-1}, \theta_1\theta_2\theta_3^{-1}\theta_2^{-1}\theta_1^{-1}) \in P_1 \times P_2 \times P_1$. It is easy to see that (p_1, p_2, p_3) is determined modulo the left action of $B \times B \times B$ on (p_1, p_2, p_3) defined in (3). □

Regarding X_{w_0} as the parameter space for the flag $V_0 \subset U_1 \subset U_2 \subset V_2$, the Bott-Samelson map $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_{w_0}$ consists of discarding the auxiliary piece of data W_1. We may now compute the *exceptional* subvariety of X_{w_0} where $\mathrm{BS}_{\mathfrak{w}}$ has a fiber that consists of more than one point. Clearly given the flag $V_0 \subset U_1 \subset U_2 \subset V_2$, the vector space W_1 satisfying $W_1 \subset V_2 \cap U_2$ will be determined except for the case where $U_2 = V_2$.

Because $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_{w_0}$ is not an isomorphism, Lemma 2 fails, but since we understand the exceptional set, we may understand how to remedy it and to express E_{w_0} in terms of $E_{s_1 s_2}$.

Proposition 15 *We have*

$$E_{w_0}(g;\nu_1,\nu_2) = E_{s_1}(g;\nu_1,\nu_2) + \sum_{\gamma_3 \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)} (E_{s_1 s_2} - E_{s_1})(\iota_{\alpha_1}(\gamma_3)g;\nu_1,\nu_2).$$

Proof The element $\gamma = \theta_1\theta_2\theta_3$ has a unique factorization

$$\iota_{\alpha_1}(\gamma_1)\iota_{\alpha_2}(\gamma_2)\iota_{\alpha_1}(\gamma_3)$$

as in Lemma 2 with $\gamma_i \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F)$ except when γ lies in the exceptional subvariety. This means that $\gamma(U_2) = V_2$, that is, when $\gamma \in G_{s_1} = B \cup B s_1 B$. These correspond to the terms where $\gamma_2 \in B_{\mathrm{SL}_2}$.

These exceptional terms contribute exactly E_{s_1}. For the remaining terms, we note that

$$\sum_{\substack{\gamma_1 \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F) \\ \gamma_2 \in B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(F) \\ \gamma_2 \notin B_{\mathrm{SL}_2}}} f(\iota_{\alpha_1}(\gamma_1)\iota_{\alpha_2}(\gamma_2)g) = E_{s_1 s_2} - E_{s_1},$$

and these terms therefore contribute the second term. □

This type of analysis would in principle allow one to represent more complicated Schubert Eisenstein series by an analog of the procedure we used for $E_{s_1 s_2}$.

8 Whittaker coefficients as Demazure characters

The Casselman-Shalika formula [7] expresses the Whittaker coefficients of Eisenstein series in terms of characters of irreducible representations of the L-group, which, when $G = \mathrm{GL}_3$, we may identify with $\mathrm{GL}_3(\mathbb{C})$. Thus let $\boldsymbol{z} = \mathrm{diag}(p^{3\nu_1+3\nu_2-2}, p^{3\nu_2-1}, 1) \in \mathrm{GL}_3(\mathbb{C})$. The values of the p-adic Whittaker function associated to $E^*(g;\nu_1,\nu_2)$ are equal to the values of the irreducible characters of the L-group evaluated at $\boldsymbol{z}$.

Demazure [8] associated an operator ∂_w with every Weyl group element. These act on functions of $\boldsymbol{z}$, if λ is a dominant weight and w_0 is the long element, $\partial_{w_0}\boldsymbol{z}^\lambda$ is the character of the irreducible representation with highest weight λ. Thus these are the values of the p-adic Whittaker functions, which are the p-parts of the Whittaker coefficient of $E^*(g;\nu_1,\nu_2)$.

We can imagine a generalization of this that associates with a Schubert Eisenstein series $E^*_w(g;\nu_1,\nu_2)$ a Whittaker coefficient that is multiplicative, and whose p-part is related to the Demazure character $\partial_w\boldsymbol{z}^\lambda$. For this, we will switch to a classical setting.

Thus we now consider the Eisenstein series on $\mathrm{GL}_3(\mathbb{R})$. Let $f_{\nu_1,\nu_2} : \mathrm{GL}_3(\mathbb{R}) \longrightarrow \mathbb{C}$ be the function defined by

$$f_{\nu_1,\nu_2}\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} k\right) = y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2}, \qquad k \in \mathrm{SO}(3), 0 < y_i \in \mathbb{R}.$$

Then if $\mathrm{re}(\nu_1), \mathrm{re}(\nu_2) > \frac{2}{3}$ we may consider

$$E(g;\nu_1,\nu_2) = \sum_{B_{\mathrm{SL}_3}(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{Z})} f_{\nu_1,\nu_2}(\gamma g),$$

We may similarly define Schubert Eisenstein series E_w by restricting to a Schubert cell, and let

$$E_w^*(g;\nu_1,\nu_2) = \zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)E_w(g;\nu_1,\nu_2).$$

Now $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and ζ is the Riemann zeta function. We will also define, as we did on the adele group $E_{s_1s_2}^{**}(g;\nu_1,\nu_2) = \zeta^*(3\nu_1)E_{s_1s_2}(g;\nu_1,\nu_2)$.

The hypothesis that a Whittaker function can be associated with a Schubert variety X_w, and that its values are Demazure characters is valid if w is the long element in a Levi subgroup, so the first test cases will be $w = s_1s_2$ or s_2s_1 on GL_3. When $w = s_1s_2$, we will show how to associate a Whittaker function with E_w^*. We will see that it is related to the Demazure character $\partial_{s_1s_2}\boldsymbol{z}^\lambda$. Our results are suggestive but do not constitute enough data to conjecture a general such relationship. The integral that we will use to define the Whittaker function is somewhat unusual since it involves an averaging process. Moreover, the Whittaker coefficient is not simply the Demazure character but the difference of two terms, each of which is a Demazure character.

We will take the representative

$$w_0 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

for the long element of the Weyl group.

Theorem 7 *We have*

$$\int_{(\mathbb{R}/\mathbb{Z})^3} E\left(w_0\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g;\nu_1,\nu_2\right) e^{2\pi i(n_1x_1+n_2x_2)}\,dx_1\,dx_2\,dx_3 \quad =$$

$$|n_1|^{\nu_1+2\nu_2-2}|n_2|^{2\nu_1+\nu_2-2} W\left(\begin{pmatrix} n_1n_2 & & \\ & n_2 & \\ & & 1 \end{pmatrix} g\right) D(n_1,n_2;\nu_1,\nu_2)$$

where $D(n_1, n_2; \nu_1, \nu_2)$ *is the Dirichlet series*

$$4 \sum_{C_1, C_2 = 1}^{\infty} \sum_{\substack{A_1, B_1 \; mod \; C_1 \\ A_2, B_2 \; mod \; C_2 \\ \gcd(A_1, B_1, C_1) = 1 \\ \gcd(A_2, B_2, C_2) = 1 \\ A_1 C_2 + B_1 B_2 + C_1 A_2 = 0}} C_1^{-3\nu_2} C_2^{-3\nu_1} e^{2\pi i (n_1 B_1 / C_1 + n_2 B_2 / C_2)}.$$

This was proved in Bump [4], where it was also shown that the Dirichlet series $D(n_1, n_2; \nu_1, \nu_2)$ could be evaluated in terms Schur polynomials, consistent with the Casselman-Shalika formula [7]. Since we will be concerned with a modification of this result and its proof, we review the method.

Proof (sketch) It is proved in Bump [4], Chapter V, that the cosets in $B_{\mathrm{SL}_3}(\mathbb{Z}) \backslash \mathrm{SL}_3(\mathbb{Z})$ may be described as follows. Let $\gamma = (\gamma_{ij})$ be a coset representative. Let $(A_1, B_1, C_1) = (\gamma_{31}, \gamma_{32}, \gamma_{33})$ be the bottom row of γ, and

$$(A_2, B_2, C_2) = \left(\begin{vmatrix} \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{vmatrix}, \begin{vmatrix} \gamma_{23} & \gamma_{21} \\ \gamma_{33} & \gamma_{31} \end{vmatrix}, \begin{vmatrix} \gamma_{22} & \gamma_{23} \\ \gamma_{32} & \gamma_{33} \end{vmatrix} \right).$$

Then $\gcd(A_1, B_1, C_1) = \gcd(A_2, B_2, C_2) = 1$, $A_1 C_2 + B_1 B_2 + C_1 A_2 = 0$, and moreover given any integer solutions $A_1, B_1, C_1, A_2, B_2, C_2$ to these relations there exists a unique coset in $B_{\mathrm{SL}_3}(\mathbb{Z}) \backslash \mathrm{SL}_3(\mathbb{Z})$. We will assume that γ is in the big Bruhat cell, since otherwise it does not contribute to the Whittaker coefficient. Then there exists $u \in U_{\mathrm{SL}_3}(\mathbb{Q})$ such that

$$u\gamma = \begin{pmatrix} C_2^{-1} & & \\ & C_2 C_1^{-1} & \\ & & C_1 \end{pmatrix} \begin{pmatrix} 1 & & \\ -B_2/C_2 & 1 & \\ A_1/C_1 & B_1/C_1 & 1 \end{pmatrix}.$$

Substituting the definition of the Eisenstein series, the integral equals

$$\sum_{C_1, C_2 \in \mathbb{Z}} \sum_{\substack{A_1, B_1 \\ A_2, B_2 \\ \gcd(A_1, B_1, C_1) = 1 \\ \gcd(A_2, B_2, C_2) = 1 \\ A_1 C_2 + B_1 B_2 + C_1 A_2 = 0}} |C_1|^{-3\nu_2} |C_2|^{-3\nu_2} \int_{(\mathbb{R}/\mathbb{Z})^3}$$

$$f\left(\begin{pmatrix} 1 & & \\ -B_2/C_2 & 1 & \\ A_1/C_1 & B_1/C_1 & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g \right) e^{2\pi i (n_1 x_1 + n_2 x_2)} dx_1 \, dx_2 \, dx_3.$$

We may telescope the summation with the integration, after which A_1, B_1 are modulo C_1 and A_2, B_2 are modulo C_2, and making a variable change we obtain $D(n_1, n_2; \nu_1, \nu_2)$ except with $e^{2\pi i(-n_1 B_1/C_2 + n_2 B_2/C_2)}$. For aesthetic purposes we may then change $A_1 \to -A_1$, $B_1 \to -B_1$ and $A_2 \to -A_2$ to obtain the stated form. Similarly we may assume that $C_i > 0$ because the sum is unchanged by replacing C_i by $|C_i|$. This accounts for the factor of 4. The Whittaker integral

$$\int_{\mathbb{R}^3} f\left(w_0 \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g\right) e^{2\pi i(n_1 x_1 + n_2 x_2)} dx_1\, dx_2\, dx_3$$

equals

$$|n_1|^{\nu_1+2\nu_2-2}|n_2|^{2\nu_1+\nu_2-2} W\left(\begin{pmatrix} n_1 n_2 & & \\ & n_1 & \\ & & 1 \end{pmatrix} g\right).$$

□

Let us consider how this must be modified when E is replaced by $E_{s_1 s_2}$. First, note that $\gamma \in B\backslash G$ is in the $s_1 s_2$ Schubert cell if and only if $\gamma_{31} = 0$; this is denoted A_1. So in this classical setting we may write

$$E_{s_1 s_2}(g) = \sum_{\substack{A_1, B_1 \\ A_2, B_2 \\ \gcd(B_1, C_1) = 1 \\ \gcd(A_2, B_2, C_2) = 1 \\ B_1 B_2 + C_1 A_2 = 0}} f(\gamma g).$$

We would like to consider

$$\int_{-\infty}^{\infty} dx_3 \int_{\mathbb{R}/\mathbb{Z}} dx_1 \int_{\mathbb{R}/\mathbb{Z}} dx_2\ E_{s_1 s_2}\left(w_0 \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g; \nu_1, \nu_2\right) e^{2\pi i(n_1 x_1 + n_2 x_2)}.$$

However this is not well-defined since the integral is not periodic for x_2. Nevertheless (assuming the Eisenstein series is in the region of absolute convergence) it is a limit of periodic functions, and we will see that we can evaluate

$$\lim_{N \longrightarrow \infty} \frac{1}{N} \int_{-\infty}^{\infty} dx_3 \int_{\mathbb{R}/\mathbb{Z}} dx_1 \int_0^N dx_2\, E_{s_1 s_2}\left(w_0 \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g; \nu_1, \nu_2\right) e^{2\pi i(n_1 x_1 + n_2 x_2)}.$$

We may follow the proof of Theorem 7. We obtain

$$\lim_{N\longrightarrow\infty}\frac{1}{N}\int_{-\infty}^{\infty}dx_3\int_{\mathbb{R}/\mathbb{Z}}dx_1\int_0^N dx_2 \sum_{C_1,C_2\in\mathbb{Z}} \sum_{\substack{B_1,A_2,B_2\\ \gcd(B_1,C_1)=1\\ \gcd(A_2,B_2,C_2)=1\\ B_1B_2+C_1A_2=0}} |C_1|^{-3\nu_2}|C_2|^{-3\nu_1}$$

$$f\left(\begin{pmatrix}1&&\\-B_2/C_2&1&\\&B_1/C_1&1\end{pmatrix}w_0\begin{pmatrix}1&x_1&x_3\\&1&x_2\\&&1\end{pmatrix}g\right)e^{2\pi i(n_1x_1+n_2x_2)}dx_1\,dx_2\,dx_3.$$

We must be careful about telescoping the summation with the integration. If we change the x matrix by an integer matrix, this means that we are making a variable change

$$\begin{pmatrix}1&&\\-B_2/C_2&1&\\&B_1/C_1&1\end{pmatrix}\longrightarrow\begin{pmatrix}1&&\\-B_2/C_2&1&\\&B_1/C_1&1\end{pmatrix}\begin{pmatrix}1&&\\k_2&1&\\k_3&k_1&1\end{pmatrix}$$

with k_i in $\mathbb{Z}$. This replaces

$$(B_1,C_1)\longrightarrow(B_1+k_1C_1,C_1)$$

and

$$(A_2,B_2,C_2)\longrightarrow(A_2-k_1B_2+(k_1k_2-k_3)C_2,B_2-k_2C_2,C_2).$$

However we may only make this variable change if it does not take us out of the s_1s_2 Schubert cell, so we require $B_1k_2+C_1k_3=0$. Since $\gcd(B_1,C_1)=1$ we require $C_1|k_2$. Hence if we collect the terms with fixed C_1, the x_2 integral is periodic with period N provided $C_1|N$. Restricting ourselves to these terms, we may then collapse the summation and the integration, and in the limit as $N\longrightarrow\infty$ we obtain

$$4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(w_0\begin{pmatrix}1&x_1&x_3\\&1&x_2\\&&1\end{pmatrix}g\right)e^{2\pi i(n_1x_1+n_2x_2)}dx_1\,dx_2\,dx_3$$

times the Dirichlet series

$$D_{s_1s_2}(n_1,n_2;\nu_1,\nu_2)=\sum_{C_1,C_2=1}^{\infty}\frac{1}{C_1}\sum_{\substack{B_1\bmod C_1\\ B_2\bmod C_1C_2\\ \gcd(B_1,C_1)=1\\ \gcd(A_2,B_2,C_2)=1\\ B_1B_2+C_1A_2=0}}|C_1|^{-3\nu_2}|C_2|^{-3\nu_1}e^{2\pi i(n_1B_1/C_1+n_2B_2/C_2)}$$

We must have $C_1|B_2$. Let us write $B_2' = B_2/C_1$. Then

$$D_{s_1s_2}(n_1,n_2;\nu_1,\nu_2) = \sum_{C_1,C_2=1}^{\infty} \frac{1}{C_1} \sum_{\substack{B_1 \bmod C_1 \\ B_2' \bmod C_2 \\ \gcd(B_1,C_1)=1 \\ \gcd(B_2',C_2)=1}} |C_1|^{-3\nu_2}|C_2|^{-3\nu_1} e^{2\pi i(n_1B_1/C_1+n_2C_1B_2'/C_2)}.$$

Let us write

$$R_{s_1s_2}(C_1,C_2,n_1,n_2) = \frac{1}{C_1} \sum_{\substack{B_1 \bmod C_1 \\ B_2' \bmod C_2 \\ \gcd(B_1,C_1)=1 \\ \gcd(B_2',C_2)=1}} e^{2\pi i(n_1B_1/C_1+n_2C_1B_2'/C_2)}.$$

Let

$$R^{**}_{s_1s_2}(C_1,C_2;n_1,n_2) = \sum_{d_2|C_2} R_{s_1s_2}(C_1,d_2,n_1,n_2).$$

Note that $R^{**}_{s_1s_2}$ is the coefficient we would obtain if we replaced $E_{s_1s_2}$ by $E^{**}_{s_1s_2} = \zeta(3\nu_1)E_{s_1s_2}$, which is the minimally normalized Schubert Eisenstein series. We will also denote $D^{**}_{s_1s_2}(n_1,n_2;\nu_1,\nu_2) = \zeta(3\nu_1)D_{s_1s_2}(n_1,n_2;\nu_1,\nu_2)$.

Proposition 16 *If* $\gcd(C_1C_2, C_1'C_2') = 1$ *then*

$$R_{s_1s_2}(C_1C_1', C_2C_2', n_1,n_2) = R_{s_1s_2}(C_1,C_2,n_1,n_2)R_{s_1s_2}(C_1',C_2',n_1,n_2)$$

and

$$R^{**}_{s_1s_2}(C_1C_1', C_2C_2', n_1,n_2) = R^{**}_{s_1s_2}(C_1,C_2,n_1,n_2)R^{**}_{s_1s_2}(C_1',C_2',n_1,n_2)$$

Proposition 17 *If* $\gcd(m_1m_2, C_1C_2) = 1$ *then*

$$R_{s_1s_2}(C_1,C_2,m_1n_1,m_2n_2) = R_{s_1s_2}(C_1,C_2,n_1,n_2).$$

The same is true for $R^{**}_{s_1s_2}$. The last two Propositions reduce the computation of $R_{s_1s_2}$ or $R^{**}_{s_1s_2}$ to the case where C_1, C_2, n_1 and n_2 are all powers of the same prime p.

Proposition 18 *We have*

$$p^{k_1} \cdot R^{**}_{s_1s_2}(p^{k_1},p^{k_2};p^{l_1},p^{l_2}) = \begin{cases} p^{k_1+k_2-1}(p-1) & \text{if } l_1 \geqslant k_1 \geqslant 1, l_2+k_1-k_2 \geqslant 0 \\ p^{k_2} & \text{if } k_1 = 0, l_2+k_1-k_2 \geqslant 0 \\ -p^{k_1-1}p^{k_2} & \text{if } l_1+1 = k_1, l_2+k_1-k_2 \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof The left-hand side equals

$$
\begin{aligned}
\sum_{h=0}^{k_2} R_{s_1 s_2}(p^{k_1}, p^{k_2-h}; p^{l_1}, p^{l_2}) &= \\
p^{k_1} \sum_{\substack{B_1 \bmod p^{k_1} \\ \mathrm{ord}(B_1)=0}} e\left(\frac{p^{l_1}B_1}{p^{k_1}}\right) \sum_{B_2 \bmod p^{k_2}} e\left(\frac{p^{l_2+k_1}B_2}{p^{k_2}}\right) &= \\
p^{k_1} \left[\sum_{\substack{B_1 \bmod p^{k_1} \\ \mathrm{ord}(B_1)=0}} e\left(\frac{p^{l_1}B_1}{p^{k_1}}\right) \right] \begin{cases} p^{k_2} & \text{if } l_2+k_1 \geqslant k_2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

and since the sum in brackets equals

$$
\left\{ \begin{array}{ll} p^{k_1} & \text{if } l_1 \geqslant k_1 \\ 0 & \text{otherwise} \end{array} \right\} - \left\{ \begin{array}{ll} p^{k_1-1} & \text{if } l_1 \geqslant k_1 - 1 \\ 0 & \text{otherwise} \end{array} \right\}
$$

the statement follows. □

It seems possible that in general Schubert Eisenstein series have Whittaker functions that can be expressed in terms of Demazure characters.

We have

$$
D^{**}_{s_1 s_2}(n_1, n_2; \nu_1, \nu_2) = \prod_p D^{**}_{s_1 s_2, p}(p^{\mathrm{ord}_p(n_1)}, p^{\mathrm{ord}_p(n_2)}; \nu_1, \nu_2)
$$

where

$$
D^{**}_{s_1 s_2, p}(p^{l_1}, p^{l_2}; \nu_1, \nu_2) = \sum_{k_1, k_2} R^{**}_{s_1 s_2}(p^{k_1}, p^{k_2}; \nu_1, \nu_2) p^{k_1(1-3\nu_2)} p^{-k_2 3\nu_1}.
$$

These p-parts may be expressed more simply as follows. Let

$$
\begin{aligned}
p^{k_1} \cdot r^{**}_{s_1 s_2}(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2}) &= \\
\begin{cases} p^{k_1+k_2} & \text{if } l_1 \geqslant k_1, l_2 + k_1 - k_2 \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Let

$$
d^{**}_{s_1 s_2, p}(p^{l_1}, p^{l_2}; \nu_1, \nu_2) = \sum_{k_1, k_2} r^{**}_{s_1 s_2}(p^{k_1}, p^{k_2}; \nu_1, \nu_2) p^{k_1(1-3\nu_2)} p^{-3k_2\nu_1}.
$$

Proposition 19 *We have*

$$D^{**}_{s_1s_2}(n_1,n_2;\nu_1,\nu_2)=d^{**}_{s_1s_2,p}(p^{l_1},p^{l_2};\nu_1,\nu_2)-p^{-3\nu_2}d^{**}_{s_1s_2,p}(p^{l_1},p^{l_2+1};\nu_1,\nu_2).$$

Proof This follows from the formula

$$p^{k_1}R^{**}_{s_1s_2}(p^{k_1},p^{k_2};p^{l_1},p^{l_2}) = \begin{cases} p^{k_1}r^{**}_{s_1s_2}(p^{k_1},p^{k_2};p^{l_1},p^{l_2})-p^{k_1-1}r^{**}_{s_1s_2}(p^{k_1-1},p^{k_2};p^{l_1},p^{l_2+1}) & \text{if } k_1>0,\\ p^{k_1}r^{**}_{s_1s_2}(p^{k_1},p^{k_2};p^{l_1},p^{l_2+1}) & \text{if } k_1=0.\end{cases}$$

□

To identify $d^{**}_{s_1s_2}$ as a Demazure character, let us identify the GL_3 weight lattice Λ with $\mathbb{Z}^3$. The simple positive roots are $\alpha_1=(1,-1,0)$ and $\alpha_2=(0,1,-1)$. The simple reflections are $s_1(\lambda_1,\lambda_2,\lambda_3)=(\lambda_2,\lambda_1,\lambda_3)$ and $s_2(\lambda_1,\lambda_2,\lambda_3)=(\lambda_1,\lambda_3,\lambda_2)$. We will associate with a weight $\lambda\in\Lambda$ with a Laurent monomial $\boldsymbol{z}^\lambda=z_1^{\lambda_1}z_2^{\lambda_2}z_3^{\lambda_3}$. If $f(\boldsymbol{z})$ is any function of the parameter $\boldsymbol{z}$ define

$$\partial_i f(\boldsymbol{z})=\partial_{s_i}f(\boldsymbol{z})=\frac{f(\boldsymbol{z})-\boldsymbol{z}^{-\alpha_i}f(s_i\boldsymbol{z})}{1-\boldsymbol{z}^{-\alpha_i}}.$$

Then it is known that the Demazure operators satisfy the braid relation $\partial_1\partial_2\partial_1=\partial_2\partial_1\partial_2$, and so if $w\in W$ we may write $\partial_w=\partial_{s_{i_1}}\cdots\partial_{s_{i_k}}$ where $w=s_{i_1}\cdots s_{i_k}$ is a reduced word representing w. If $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ then λ is dominant if $\lambda_1\geqslant\lambda_2\geqslant\lambda_3$. In this case, it is known that $\partial_{w_0}\boldsymbol{z}^\lambda$ is the character of the irreducible representation of $GL_3(\mathbb{C})$ with highest weight λ, evaluated at $\boldsymbol{z}=\mathrm{diag}(z_1,z_2,z_3)$.

Let $\boldsymbol{z}=\mathrm{diag}(p^{3\nu_1+3\nu_2-2},p^{3\nu_2-1},1)\in GL_3(\mathbb{C})$. These are the Langlands parameters of the Eisenstein series. Let $\lambda=(l_1+l_2,l_1,0)$.

Theorem 8 *We have*

$$d^{**}_{s_1s_2,p}(p^{l_1},p^{l_2};\nu_1,\nu_2)=\boldsymbol{z}^{-\lambda}\partial_{s_1s_2}\boldsymbol{z}^\lambda.$$

Proof We have

$$\boldsymbol{z}^{-\lambda}\partial_2\boldsymbol{z}^\lambda=\sum_{k_1=0}^{l_1}\boldsymbol{z}^{(l_1+l_2,l_1-k_1,k_1)},\qquad \partial_1\partial_2\boldsymbol{z}^\lambda=\sum_{k_1=0}^{l_1}\sum_{k_2=0}^{l_2+k_1}\boldsymbol{z}^{(l_1+l_2-k_2,l_1+k_2-k_1,k_1)},$$

and so

$$\boldsymbol{z}^{-\lambda}\partial_{s_1s_2}\boldsymbol{z}^\lambda=\sum_{k_1=0}^{l_1}\sum_{k_2=0}^{l_2+k_1}\boldsymbol{z}^{(-k_2,k_2-k_1,k_1)}=\sum_{k_1=0}^{l_1}\sum_{k_2=0}^{l_2+k_1}p^{k_1(1-3\nu_2)}p^{k_2(1-3\nu_1)}.$$

On the other hand

$$d^{**}_{s_1 s_2, p}(p^{l_1}, p^{l_2}; \nu_1, \nu_2) = \sum_{\substack{k_1 \leqslant l_1 \\ k_2 \leqslant l_2 + k_1}} p^{k_1(1-3\nu_2)} p^{k_2(1-3\nu_1)},$$

and these are the same. □

References

[1] R. Bott and H. Samelson. Applications of the theory of Morse to symmetric spaces. *Amer. J. Math.*, 80:964–1029, 1958.

[2] B. Brubaker, D. Bump, and S. Friedberg. Weyl group multiple Dirichlet series, Eisenstein series and crystal bases. *Ann. Math*, 173(2):1081–1120, 2011.

[3] B. Brubaker, D. Bump, and A. Licata. Whittaker functions and Demazure operators, *preprint*, 2011.

[4] D. Bump. *Automorphic forms on GL(3,R)*, volume 1083 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984.

[5] D. Bump and D. Goldfeld. A Kronecker limit formula for cubic fields. In *Modular forms (Durham, 1983)*, Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., pages 43–49. Horwood, Chichester, 1984.

[6] W. Casselman. The unramified principal series of p-adic groups. I. The spherical function. *Compositio Math.*, 40(3):387–406, 1980.

[7] W. Casselman and J. Shalika. The unramified principal series of p-adic groups. II. The Whittaker function. *Compositio Math.*, 41(2):207–231, 1980.

[8] M. Demazure. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup. (4)*, 7:53–88, 1974. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.

[9] S. G. Gindikin and F. I. Karpelevič. Plancherel measure for symmetric Riemannian spaces of non-positive curvature. *Dokl. Akad. Nauk SSSR*, 145:252–255, 1962.

[10] H. Jacquet. Fonctions de Whittaker associées aux groupes de Chevalley. *Bull. Soc. Math. France*, 95:243–309, 1967.

[11] H. Jacquet, I. Piatetski-Shapiro, and J. Shalika. Automorphic forms on GL(3). I. *Ann. of Math. (2)*, 109(1):169–212, 1979.

[12] I. I. Pjateckij-Šapiro. Euler subgroups. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 597–620. Halsted, New York, 1975.

[13] A. I. Vinogradov and L. A. Tahtadžjan. Theory of the Eisenstein series for the group SL(2,R) and its application to a binary problem. I. Fourier expansion of the highest Eisenstein series. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 76:5–52, 216, 1978. Analytic number theory and the theory of functions.