COUNTING SMOOTH SOLUTIONS TO THE EQUATION $A+B=C$

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Abstract. This paper studies integer solutions to the *abc* equation $A+B=C$ in which none of A, B, C have a large prime factor. We set $H(A,B,C) = \max(|A|,|B|,|C|)$, and consider primitive solutions ($\gcd(A,B,C)=1$) having no prime factor p larger than $(\log H(A,B,C))^{\kappa}$, for a given finite κ. On the assumption that the Generalized Riemann hypothesis (GRH) holds, we show that for any $\kappa > 8$ there are infinitely many such primitive solutions having no prime factor larger than $(\log H(A,B,C))^{\kappa}$. We obtain in this range an asymptotic formula for the number of such suitably weighted primitive solutions.

1. Introduction

A recurring topic of investigation in number theory is the relation between additive and multiplicative structures of integers. A celebrated example is the *abc*-conjecture of Masser [25] and Oesterlé [26], cf. [4, Chap. 12]. In its weak form, the *abc*-conjecture asserts that there is a constant $\kappa_1 > 0$ such that for any $\epsilon > 0$ there are only finitely many solutions to the equation $A+B=C$ with $ABC \neq 0$, g.c.d. $(A,B,C)=1$ and such that

$$\max(|A|,|B|,|C|) \leq \Big(\prod_{p|ABC} p \Big)^{\kappa_1 - \epsilon}.$$

One may construct examples to show that κ_1, if it exists, cannot be smaller than 1 (see Stewart and Tijdeman [27]), and the strong form of the *abc*-conjecture postulates that in fact $\kappa_1 = 1$ is permissible. In this paper, we study a different statistic related to the prime factorization of ABC. In place of the *radical* $\prod_{p|ABC} p$, we study the *smoothness* $\max_{p|ABC} p$. In [24] we formulated the following conjecture, which we term the *xyz* conjecture.

xyz conjecture (weak form). *There exists a constant* $\kappa_0 > 0$ *such that the following hold.*

(a) For each $\epsilon > 0$ *there are only finitely many solutions* (X,Y,Z) *to the equation* $X+Y=Z$ *with g.c.d*$(X,Y,Z)=1$ *and*

$$\max_{p|XYZ} p < (\log \max(|X|,|Y|,|Z|))^{\kappa_0 - \epsilon}. \tag{1.1}$$

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(b) For each $\epsilon > 0$ there are infinitely many solutions (X, Y, Z) to the equation $X + Y = Z$ with g.c.d.$(X, Y, Z) = 1$ and

$$\max_{p|XYZ} p < (\log \max(|X|, |Y|, |Z|))^{\kappa_0+\epsilon}. \tag{1.2}$$

We shall call a solution $A + B = C$ primitive if g.c.d.$(A, B, C) = 1$. The restriction to primitive solutions in the *abc* and *xyz* conjectures is needed to exclude examples like $a + a = 2a$ where a is a high perfect power, or a is very smooth.

For any primitive solution (X, Y, Z) to $X+Y = Z$ we define its *smoothness exponent* $\kappa_0(X, Y, Z)$ by

$$\kappa_0(X, Y, Z) := \frac{\log \max_{p|XYZ} p}{\log \log \max(|X|, |Y|, |Z|)}. \tag{1.3}$$

Our interest then is in the xyz-smoothness exponent κ_0 which is defined as the lim inf of $\kappa_0(X, Y, Z)$ as $\max(|X|, |Y|, |Z|) \to \infty$. *A priori* we have $0 \le \kappa_0 \le +\infty$, and the weak form of the xyz-conjecture asserts that it is positive and finite. We next give a heuristic for the weak xyz conjecture which also suggests a plausible value for κ_0.

xyz conjecture (strong form). *The xyz-smoothness exponent κ_0 equals 3/2.*

A natural number n is said to be y-smooth if all its prime factors lie below y. Throughout we shall let $\mathcal{S}(y)$ denote the set of y-smooth numbers, and $\Psi(x, y)$ shall count the number of positive integers below x lying in $\mathcal{S}(y)$.

Consider all the triples $(X, Y, -Z)$ drawn from the interval $[1, H]$ but restricted to having all prime factors smaller than $(\log H)^\kappa$. We wish to find solutions to $X + Y - Z = 0$. There are $\Psi(H, (\log H)^\kappa)^3$ such triples, each having a sum $X + Y - Z$ that falls in the interval $[-H, 2H]$. If these sums were randomly distributed, the chance that the value 0 is hit might be expected to be approximately proportional to

$$P(H, \kappa) := \frac{\Psi(H, (\log H)^\kappa)^3}{H}. \tag{1.4}$$

It is known that (see (4.5) below) for fixed $\kappa > 1$, one has

$$\Psi(x, (\log x)^\kappa) = x^{1-\frac{1}{\kappa}+o(1)}, \tag{1.5}$$

as $x \to \infty$. Thus for $\kappa > 1$ the number of such triples (X, Y, Z) is at most $\Psi(H, (\log H)^\kappa)^3 = H^{3(1-\frac{1}{\kappa}+o(1))}$, and if $\kappa < \frac{3}{2}$ this is $< H^{1-\epsilon}$, so that $P(H, \kappa) = H^{-\epsilon}$. This leads us to believe that $\kappa_0 \ge \frac{3}{2}$.

We derive a matching heuristic lower bound for the number of relatively prime triples. Take X to be a number composed of exactly $K := [\log H/(\kappa \log\log H)]$ distinct primes all below $(\log H)^\kappa$. Using Stirling's formula there are $\binom{\pi((\log H)^\kappa)}{K} = H^{1-1/\kappa+o(1)}$ such values of X all lying below H. Given X, choose Y to be a number composed of exactly K distinct primes below $(\log H)^\kappa$, but avoiding the primes dividing X. There are $\binom{\pi((\log H)^\kappa)-K}{K} = H^{1-1/\kappa+o(1)}$ such values of Y. Finally choose Z to be a number composed of exactly K distinct primes below $(\log H)^\kappa$ avoiding the primes dividing X and Y. There are $\binom{\pi((\log H)^\kappa)-2K}{K} = H^{1-1/\kappa+o(1)}$ such values of Z. We conclude therefore that there are at least $H^{3-3/\kappa+o(1)}$ such triples, and hence we expect that $\kappa_0 \le \frac{3}{2}$.

In Theorem 1.1 of [24] we observed that lower bounds for the xyz smoothness exponent follow from the *abc* conjecture.

Proposition 1.1. *The weak form of the abc-conjecture implies that the xyz smoothness exponent satisfies $\kappa_0 \ge \kappa_1$. In particular, the strong form of the abc-conjecture implies that $\kappa_0 \ge 1$.*

It is interesting to note that even the strong form of the *abc* conjecture is insufficient to imply the conjectured lower bound $\kappa_0 \ge \frac{3}{2}$ above on the xyz exponent.

This paper studies the upper bound part of the xyz-conjecture. Assuming the truth of the *Generalized Riemann Hypothesis (GRH)*, which states that all non-trivial zeros of the Riemann zeta function and Dirichlet L-functions lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, we shall show that $\kappa_0 \le 8$.

Theorem 1.2. *Assume the truth of the Generalized Riemann Hypothesis (GRH). Then for each $\epsilon > 0$ there are infinitely many primitive solutions (X, Y, Z) to $X + Y = Z$ such that all the primes dividing XYZ are smaller than $(\log \max(|X|, |Y|, |Z|))^{8+\epsilon}$. In other words, $\kappa_0 \le 8$.*

This result is an immediate consequence of the following stronger result, which gives a lower bound for the number of primitive solutions in this range.

Theorem 1.3. (Counting Primitive Smooth Solutions) *Assume the truth of the Generalized Riemann Hypothesis (GRH). Then for each fixed $\kappa > 8$ the number of primitive integer solutions $N^*(H,\kappa)$ to $X+Y=Z$ with $0 \le X, Y, Z \le H$ and such that the largest prime factor of XYZ is $< (\log H)^\kappa$ satisfies*

(1.6)
$$N^*(H,\kappa) \ge \mathfrak{S}_\infty\Big(1-\frac{1}{\kappa}\Big)\mathfrak{S}_f^*\Big(1-\frac{1}{\kappa}, (\log H)^\kappa\Big)\frac{\Psi(H,(\log H)^\kappa)^3}{H}(1+o(1)),$$

as $H \to \infty$. Here the "archimedean singular series" (more properly, "singular integral") $\mathfrak{S}_\infty(c)$ is defined, for $c > \frac{1}{3}$, by

$$\mathfrak{S}_\infty(c) := c^3 \int_0^1 \int_0^{1-t_1} (t_1 t_2 (t_1+t_2))^{c-1} dt_1 dt_2, \tag{1.7}$$

and the "primitive non-archimedean singular series" $\mathfrak{S}_f^*(c,y)$ is defined by

(1.8)
$$\mathfrak{S}_f^*(c,y) := \prod_{p\le y}\Big(1+\frac{p-1}{p(p^{3c-1}-1)}\Big(\frac{p-p^c}{p-1}\Big)^3\Big)\Big(1-\frac{1}{p^{3c-1}}\Big)\prod_{p>y}\Big(1-\frac{1}{(p-1)^2}\Big).$$

We expect that the lower bound given by the right side of (1.6) should give an asymptotic formula for $N^*(H,\kappa)$ in this range of κ, and that proving this should be accessible by elaboration of the methods of this paper. The estimate (1.6) is in accordance with the heuristic (1.4) which would have predicted a main term of $\Psi(H,(\log H)^\kappa)^3/H$. In the range $\kappa > 8$ we see that the main term in (1.6) differs from the heuristic only by the factor $\mathfrak{S}_\infty(1-1/\kappa)\mathfrak{S}_f^*(1-1/\kappa,(\log H)^\kappa)$. An argument below shows this factor is bounded away from 0 and ∞, and for fixed κ it approaches a constant (depending on κ) as $H\to\infty$. As $\kappa\to\infty$, this constant factor approaches $\frac{1}{2}$, and the main term $\frac{1}{2}\Psi(H,(\log H)^\kappa)^3/H$ is the expected number of solutions to $X+Y=Z$ when X, Y and Z are drawn from a random subset of $[1,H]$ with cardinality $\Psi(H,(\log H)^\kappa)$. Thus our heuristic is very accurate in the range $\kappa\to\infty$.

The "main term" on the right side of (1.6) is well-defined in the range $\kappa > \frac{3}{2}$ where the heuristic above is expected to apply. Here $\kappa>\frac{3}{2}$ corresponds to $c>\frac{1}{3}$, and the "archimedean singular integral" (1.7) defines an analytic function on the half-plane $\mathrm{Re}(c) > \frac{1}{3}$ which diverges at $c=\frac{1}{3}$, while the "non-archimedean singular series" $\mathfrak{S}_f^*(c,y)$ is well-defined for all $c>0$. The archimedean singular series is uniformly bounded on any half-plane $\mathrm{Re}(c) > \frac{1}{3}+\epsilon$. For the non-archimedean singular series, we find that its limiting behavior as $y=(\log H)^\kappa\to\infty$ changes at the threshold value $\kappa=2$, corresponding to $c=\frac{1}{2}$. Namely, one has

$$\lim_{H\to\infty}\mathfrak{S}_f^*(1-\frac{1}{\kappa},(\log H)^\kappa) = \begin{cases} \mathfrak{S}_f^*(1-\frac{1}{\kappa}) & \text{for } \kappa>2,\\ 0 & \text{for } 0<\kappa\le 2,\end{cases} \tag{1.9}$$

where for $c>\frac{1}{2}$ we set

$$\mathfrak{S}_f^*(c) := \prod_p \Big(1+\frac{1}{p^{3c-1}}\Big(\frac{p-1}{p}\Big(\frac{p-p^c}{p-1}\Big)^3-1\Big)\Big). \tag{1.10}$$

(This follows from (1.8)). The Euler product (1.10) converges absolutely and defines an analytic function $\mathfrak{S}_f^*(c)$ on the half-plane $\mathrm{Re}(c)>\frac{1}{2}$; this function is uniformly bounded on any half-plane $\mathrm{Re}(c)>\frac{1}{2}+\epsilon$, Furthermore for values corresponding the $2\le\kappa<\infty$ (i.e. $\frac{1}{2}<c<1$) the "non-archimedean singular series" $\mathfrak{S}_f(c,y)$ remains bounded away from 0. We conclude that for $2<\kappa<\infty$ the "main term" estimate for $N^*(H,\kappa)$ agrees with the prediction of the heuristic argument given earlier. In the region $1<\kappa\le 2$, although (1.10) gives $\mathfrak{S}_f^*(1-\frac{1}{\kappa},(\log H)^\kappa)\to 0$ as $H\to\infty$, nevertheless one can show

$$\mathfrak{S}_f^*(1-\frac{1}{\kappa},(\log H)^\kappa) \gg \exp(-(\log H)^{2-\kappa}). \tag{1.11}$$

This bound implies that $\mathfrak{S}_f^*(1-\frac{1}{\kappa},(\log H)^\kappa) \gg H^{-\epsilon}$ for any $\epsilon>0$. A consequence is that for $\frac{3}{2}<\kappa\le 2$ the "main term" on the right side of (1.7) is still of the same order $H^{2-\frac{3}{\kappa}+o(1)}$ as the heuristic predicts. Thus it could still be the case that this "main term" gives a correct order of magnitude estimate for $N^*(H,\kappa)$ even in this range.

Next we compare the number $N^*(H,\kappa)$ of primitive smooth solutions with the total number $N(H,\kappa)$ of smooth solutions below H. Now $N(H,\kappa)$ already has a contribution coming from smooth multiples of the solution $(X,Y,Z)=(1,1,2)$ that gives

$$N(H,\kappa) \geq \Psi(\tfrac{1}{2}H, (\log H)^{\kappa}) \geq H^{1-\frac{1}{\kappa}+o(1)}, \text{ as } H\to\infty. \tag{1.12}$$

For $1 \leq \kappa < 2$ this lower bound exceeds the heuristic estimate $H^{2-\frac{3}{\alpha}+o(1)}$ for $N^*(H,\kappa)$ by a positive power of H. It follows that the heuristic given for primitive smooth solutions should not apply to smooth solutions $N(H,\kappa)$ for $1<\kappa<2$, and furthermore it indicates that on this range the density of primitive smooth solutions in the set of all smooth solutions below H will approach zero as $H\to\infty$.

We may consider for more general κ the limiting behavior as $H\to\infty$ of the relative density of primitive smooth solutions. Here we conjecture there is a threshold value at $\kappa=3$ where this behavior changes qualitatively.

Conjecture 1. *(*Relative Density of Primitive Solutions*) There holds*

$$\lim_{H\to\infty}\frac{N^*(H,\kappa)}{N(H,\kappa)} = \begin{cases} \frac{1}{\zeta(2-\frac{3}{\kappa})}, & \text{for } 3<\kappa<\infty, \\ 0 & \text{for } 1<\kappa\leq 3. \end{cases} \tag{1.13}$$

As evidence in favor of this conjecture, Theorem 2.3 below shows, assuming GRH, that a weighted version of this conjecture holds for $\kappa>8$. Further evidence is the fact that for each $\kappa>3$ the ratios of the conjectured "main terms" in the asymptotic formulas for these quantities have the limiting value $\zeta(2-\frac{3}{\kappa})$ as $H\to\infty$, a result implied by (2.9) below. Finally, the discussion above gives support for its truth on the range $1<\kappa\leq 2$.

In §2 we describe the main technical results from which the theorems above are derived. Our main estimate (Theorem 2.1) gives an asymptotic formula with error term which counts weighted (primitive and imprimitive) integer solutions to the xyz-equation in the range $\kappa>8$. This result will be established using the Hardy-Littlewood method ([29]) combined with the Hildebrand-Tenenbaum saddle point method ([21], [19], [22]) for estimating the size of $\Psi(x,y)$. We then derive a weighted count of primitive solutions (Theorem 2.2) using inclusion-exclusion. Theorem 1.3 is deduced from Theorem 2.2. It would be interesting to see whether our main results could be made unconditional. At the moment, the best known unconditional results are due to Balog and Sarközy [2], [3] who showed (in a closely related problem) for any large N, there are solutions to $X+Y+Z=N$ with the largest prime factor of XYZ being smaller than $\exp(3\sqrt{\log N\log\log N})$.

Our problem may also be viewed as a special case of the S-unit equation. Given a finite set of primes S, one can consider relatively prime solutions to the S-unit equation $X + Y = Z$ where all prime factors of XYZ are in the set S. In 1988 Erdős, Stewart and Tijdeman [12] showed the existence of collections of primes S with $|S| = s$ such that the S-unit equation $X+Y = Z$ has "exponentially many" solutions, namely at least $\exp((4-\epsilon)s^{\frac{1}{2}}(\log s)^{-\frac{1}{2}})$ solutions, for $s \geq s_0(\epsilon)$ sufficiently large. Recently Konyagin and the second author [23] improved this construction, to show that there exist S such that the S-unit equation has at least $\exp(s^{2-\sqrt{2}-\epsilon})$ solutions. In the other direction, Evertse [13, Theorem 1] has shown that the number of solutions to the S-unit equation is at most $3 \times 7^{2s+3}$.

In the constructions above the sets of primes S were tailored to have large numbers of solutions. However the simplest set of such primes to consider is the initial segment of primes $S = \mathcal{P}(y) := \{p : p \text{ prime }, p \leq y\}$. Erdős, Stewart and Tijdeman conjectured ([12, p. 49, top]) that a similar property should hold in this case, asserting that for $s = |S|$ and each $\epsilon > 0$ there should be at least $\exp(s^{\frac{2}{3}-\epsilon})$ S-unit solutions to $X + Y = Z$ and at most $\exp(s^{\frac{2}{3}+\epsilon})$ such solutions, for all $s > s_0(\epsilon)$. Their conjecture was motivated by a heuristic similar to the one given above for the strong xyz-conjecture.

As an easy consequence of Theorem 1.3 we deduce, conditional on GRH, a weak form of this conjecture, at the end of §2.

Theorem 1.4. *Assume the truth of the Generalized Riemann Hypothesis (GRH). Let S denote the first s primes, and let $N(S)$ count the number of primitive solutions (X, Y, Z) to the S-unit equation $X + Y = Z$. Then for each $\epsilon > 0$, we have $N(S) \gg_\epsilon \exp(s^{\frac{1}{8}-\epsilon})$.*

The approach in this paper will apply to other linear additive problems involving smooth numbers. For instance, one can treat smooth solutions of homogeneous linear ternary Diophantine equations $aX + bY + cZ = 0$ with arbitrary integer coefficients (a, b, c). One may also impose congruence side conditions on the prime factors allowed, for example smooth solutions with all prime factors $p \equiv 1 \mod 4$. In this situation there may occur local congruence obstructions to existence of solutions, and naturally the singular series must be modified to take such features into account. It would also be of interest to extend the xyz-conjecture to solutions of $X+Y = Z$ in algebraic number fields, or to algebraic function fields over finite fields. Finally, it would be interesting to see if analogues of Waring's problem using very smooth numbers could be established. This has been treated by Harcos [15], who obtained unconditional results for Waring's problem in the smoothness range corresponding to the results of Balog and Sarközy mentioned earlier.

2. Counting Smooth Solutions: Main Technical Results

Let x and y be large. Our aim is to count solutions to $X + Y = Z$ with X, Y and Z being pairwise coprime y-smooth integers lying below x. We shall simplify the problem by first counting all solutions, primitive and imprimitive, to $X + Y = Z$ with X, Y and Z being y-smooth integers up to x. We shall also find it convenient to replace the sharp cut-off of being less than x by counting solutions with suitable weights approximating the sharp cut-off. Once this is achieved, a sieve argument will enable us to recover primitive solutions from all solutions.

More formally, let $\Phi(x) \in C_c^\infty(\mathbb{R}^+)$ be a smooth, compactly supported, real-valued function on the positive real axis. We shall develop first an asymptotic formula for

$$N(x,y;\Phi) := \sum_{\substack{X,Y,Z \in \mathcal{S}(y) \\ X+Y=Z}} \Phi\Big(\frac{X}{x}\Big)\Phi\Big(\frac{Y}{x}\Big)\Phi\Big(\frac{Z}{x}\Big), \tag{2.1}$$

which counts weighted primitive and imprimitive solutions.

Theorem 2.1. (Weighted Smooth Integer Solutions Count) *Assume the truth of the GRH. Let Φ be a fixed smooth, compactly supported, real valued function in $C_c^\infty(\mathbb{R}^+)$. Let x and y be large, with $(\log x)^{8+\delta} \leq y \leq \exp((\log x)^{\frac{1}{2}-\delta})$ for some $\delta > 0$. Define κ by the relation $y = (\log x)^\kappa$. Then, we have*

(2.2)

$$N(x,y;\Phi) = \mathfrak{S}_\infty\Big(1-\frac{1}{\kappa},\Phi\Big)\mathfrak{S}_f\Big(1-\frac{1}{\kappa},y\Big)\frac{\Psi(x,y)^3}{x} + O_\delta\Big(\frac{\Psi(x,y)^3}{x}\frac{\log\log y}{\log y}\Big).$$

Here the "archimedean singular series" $\mathfrak{S}_\infty(c,\Phi)$ is given by

$$\mathfrak{S}_\infty(c,\Phi) := c^3 \int_0^\infty \int_0^\infty \Phi(t_1)\Phi(t_2)\Phi(t_1+t_2)\,(t_1 t_2(t_1+t_2))^{c-1}\,dt_1 dt_2, \tag{2.3}$$

and the "non-archimedean singular series" $\mathfrak{S}_f$ is defined by

$$\mathfrak{S}_f(c,y) = \prod_{p\leq y}\Big(1+\frac{p-1}{p(p^{3c-1}-1)}\Big(\frac{p-p^c}{p-1}\Big)^3\Big)\prod_{p>y}\Big(1-\frac{1}{(p-1)^2}\Big). \tag{2.4}$$

In our proof, it is convenient to restrict Φ to be compactly supported away from 0. This restriction prevents us from obtaining an asymptotic formula for the number of nonnegative solutions to $X+Y=Z$ with $Z \leq x$ and XYZ being y-smooth, which corresponds to choosing Φ to be the characteristic function $\chi_{[0,1]}$ of the interval $[0,1]$. We do expect that the asymptotic formula given in Theorem 2.1 will continue to hold in this case. In any event this result suffices to obtain a lower bound for this number of solutions by choosing a smooth function Φ compactly supported inside $\mathbb{R}^+$ which minorizes the characteristic function of $[0,1]$.

The compact support of $\Phi(x)$ guarantees that the "weighted archimedean singular series" $\mathfrak{S}_\infty(c,\Phi)$ is defined for all real c. In contrast the "non-archimedean singular series" $\mathfrak{S}_f(c,y)$ is given by an Euler product that converges to an analytic function for $\mathrm{Re}(c) > \frac{1}{3}$ and diverges at $c=\frac{1}{3}$; here individual terms in this Euler product diverge at $c=\frac{1}{3}$. We observe also that $\mathfrak{S}_f(c,y)$ has a phase change in its behavior as $y\to\infty$ at the threshold value $c = \frac{2}{3}$ corresponding to $\kappa = 3$. Namely, we have

$$\lim_{y\to\infty} \mathfrak{S}_f(1 - \frac{1}{\kappa}, y) = \begin{cases} \mathfrak{S}_f(1-\frac{1}{\kappa}) & \text{for } \kappa > 3, \\ +\infty & \text{for } 0 < \kappa \le 3, \end{cases} \tag{2.5}$$

where for $c > \frac{2}{3}$ we define

$$\mathfrak{S}_f(c) := \prod_p \Big(1 + \frac{p-1}{p(p^{3c-1}-1)}\Big(\frac{p-p^c}{p-1}\Big)^3\Big). \tag{2.6}$$

The Euler product (2.6) converges absolutely to an analytic function of c on the half-plane $\mathrm{Re}(c) > \frac{2}{3}$, and diverges at $c = \frac{2}{3}$. Outside this half-plane, on the range $\frac{1}{2} < c \le \frac{2}{3}$, although one has $\mathfrak{S}_f(1-\frac{1}{\kappa}, y) \to \infty$ as $y \to \infty$, one can show that

$$\mathfrak{S}_f(1 - \frac{1}{\kappa}, y) \ll \exp(y^{3/\kappa - 1}).$$

A consequence is that for $2 < \kappa \le 3$ one has $\mathfrak{S}_f(1-\frac{1}{\kappa}, (\log H)^\kappa) \ll H^\epsilon$ for any positive ϵ, which suggests that the heuristic argument of section 1.2 may continue to apply to $N(H, \kappa)$ on this range.

Using a sieve argument together with Theorem 2.1, we shall treat the weighted count of primitive solutions:

$$N^*(x, y; \Phi) := \sum_{\substack{X,Y,Z \in \mathcal{S}(y) \\ X+Y=Z, \gcd(X,Y,Z)=1}} \Phi\Big(\frac{X}{x}\Big)\Phi\Big(\frac{Y}{x}\Big)\Phi\Big(\frac{Z}{x}\Big). \tag{2.7}$$

Theorem 2.2. (Weighted Primitive Integer Solutions Count) *Assume the truth of the GRH. Let Φ be a fixed smooth, compactly supported, real valued function in $C_c^\infty(\mathbb{R}^+)$. Let x and y be large with $(\log x)^{8+\delta} \le y \le \exp((\log x)^{\frac{1}{2}-\delta})$. Define κ by the relation $y = (\log x)^\kappa$. Then, we have*

$$N^*(x, y; \Phi) = \mathfrak{S}_\infty\Big(1 - \frac{1}{\kappa}, \Phi\Big)\mathfrak{S}_f^*\Big(1 - \frac{1}{\kappa}, y\Big)\frac{\Psi(x,y)^3}{x} + O\left(\frac{\Psi(x,y)^3}{x(\log y)^{\frac{1}{4}}}\right),$$

where the primitive non-archimedean singular series $\mathfrak{S}_f^*(c, y)$ was defined in (1.8).

Theorem 2.1 and Theorem 2.2 together imply that for nonnegative functions Φ a smoothed analogue of Conjecture 1 holds for $\kappa > 8$.

Theorem 2.3. (Relative Density of Weighted Primitive Smooth Solutions) *Assume the truth of the GRH. Then for any nonnegative function $\Phi(x) \in C_c^\infty(\mathbb{R}_{>0})$ not identically zero, there holds*

$$\lim_{x\to\infty} \frac{N^*(x, (\log x)^\kappa; \Phi)}{N(x, (\log x)^\kappa; \Phi)} = \frac{1}{\zeta(2 - \frac{3}{\kappa})}, \quad \textit{for } \kappa > 8. \tag{2.8}$$

Concerning smaller values of κ, we expect that the asymptotic formulae given in Theorem 2.1 and Theorem 2.2 continue to hold in the range $\kappa > 3$ (so that $c = 1 - 1/\kappa > 2/3$). If so, then in this range both $N(x, y; \Phi)$ and $N^*(x, y; \Phi)$ would be of comparable size, with both being of size about $\Psi(x, y)^3/x$, conforming to the heuristic (1.4). If $1/2 < c \le \frac{2}{3}$, then $\mathfrak{S}_f^*(c, y)$ is of constant size, but $\mathfrak{S}_f(c, y)$ diverges as $y \to \infty$. Thus for the corresponding range $2 < \kappa \le 3$, we might still hope that the asymptotic formulae of Theorems 2.1 and 2.2 are true, but note that in this range there are significantly fewer primitive solutions compared to imprimitive ones.

The upper bound $y \le \exp((\log x)^{\frac{1}{2}-\delta})$ imposed in proving Theorems 2.1 and 2.2 facilitates some of our calculations, but it should be possible to remove this condition entirely and obtain similar results. We have not done so, since our interest is in small values of y, and moreover in larger ranges of y one would expect an unconditional treatment by different means.

Before proceeding to discuss the proofs of our main results stated above, we show how the theorems stated in the introduction, as well as Theorem 2.3, follow from these weighted versions.

Proof of Theorem 1.3. Given any $\epsilon > 0$ we may construct a smooth function Φ_ϵ such that Φ_ϵ is smooth and supported on $[\epsilon, 1-\epsilon]$, always lies between 0 and 1, and equals 1 on the interval $[2\epsilon, 1-2\epsilon]$. Then $N^*(H, \kappa) \ge N^*(H, (\log H)^\kappa; \Phi_\epsilon)$, and we may use Theorem 2.2 to evaluate the latter quantity. Since $\mathfrak{S}_\infty(c, \Phi_\epsilon) \to \mathfrak{S}_\infty(c)$ as $\epsilon \to 0$, we deduce Theorem 1.3. $\square$

Theorem 1.2 follows immediately from Theorem 1.3.

Proof of Theorem 1.4. Let S denote the first s primes, and choose $H = \exp(s^{1/8-\epsilon})$ and $y = p_s$. Then $(\log H)^{8+\epsilon} = (s^{\frac{1}{8}-\epsilon})^{8+\epsilon} < s < y$, so that $N(S) \ge N^*(H, p_y) \ge N^*(H, (\log H)^{8+\epsilon})$. Assuming the GRH, Theorem 1.3 gives, for sufficiently large H, $N(S) \ge C_\epsilon H^{2-3/(8+\epsilon)} \ge H$, as asserted. $\square$

Proof of Theorem 2.3. This result is based on the identity of Euler products (2.9)

$$\mathfrak{S}_f^*(c) := \prod_p \left(\left(1 + \frac{p-1}{p(p^{3c-1}-1)}\left(\frac{p-p^c}{p-1}\right)^3\right)\left(1 - \frac{1}{p^{3c-1}}\right)\right) = \frac{1}{\zeta(3c-1)}\mathfrak{S}_f(c).$$

which follows taking $y \to \infty$ in (1.8). This identity shows that $\mathfrak{S}_f(c)$ has a meromorphic continuation to the half-plane $\mathrm{Re}(c) > \frac{1}{2}$, with its only singularity on this region being a simple pole at $c = \frac{2}{3}$ having residue $\frac{1}{3}\mathfrak{S}_f^*(\frac{2}{3})$. In particular, for real $c = 1 - \frac{1}{\kappa} > \frac{2}{3} + \epsilon$ we have

$$\mathfrak{S}_f(c, y) = \mathfrak{S}_f(c)\Big(1 + O_\epsilon\Big(\frac{1}{y}\Big)\Big),$$

and for real $c > \frac{1}{2} + \epsilon$ we have

$$\mathfrak{S}_f^*(c, y) = \mathfrak{S}_f^*(c)\Big(1 + O_\epsilon\Big(\frac{1}{y}\Big)\Big).$$

Substituting these estimates in the main terms of Theorem 2.1 and Theorem 2.2 yields, for $\kappa > 8+\delta$, the estimate

(2.10)
$$N^*(x,(\log x)^\kappa;\Phi) = \frac{1}{\zeta(2-\frac{3}{\kappa})} N(x,(\log x)^\kappa;\Phi)) \left(1 + O_\delta\left(\frac{1}{(\log\log x)^{\frac{1}{4}}}\right)\right).$$

The positivity hypothesis on Φ implies that $N(x,(\log x)^\kappa;\Phi) > 0$ so we may divide both sides of (2.10) by it to obtain the ratio estimate (2.8). □

We shall use the Hardy-Littlewood circle method to evaluate $N(x,y;\Phi)$. To this end, we introduce the weighted exponential sum

$$E(x,y;\alpha) := \sum_{n\in\mathcal{S}(y)} e(n\alpha)\Phi\Big(\frac{n}{x}\Big), \tag{2.11}$$

where throughout we use $e(x) := e^{2\pi i x}$. Then we have

$$N(x,y;\Phi) = \int_0^1 E(x,y;\alpha)^2 E(x,y;-\alpha)d\alpha, \tag{2.12}$$

because in multiplying out the exponential sums in the integral, only terms (n_1,n_2,n_3) with $n_1+n_2-n_3=0$ contribute. The crux of the problem then is to understand the weighted exponential sum $E(x,y;\alpha)$.

To do this, we show how to express the term $e(n\alpha)\Phi(n/x)$ in terms of sums over multiplicative Dirichlet characters to a certain modulus and integrals of n^{it} over t in a certain range. This is carried out precisely in Section 3, but the idea is implicit in the original 'Partitio Numerorum' papers of Hardy and Littlewood ([16], [17]) where they dealt with the ternary Goldbach problem assuming a weaker form of GRH. We hope that the explicit form that we give may be useful in other contexts.

The decomposition of $e(n\alpha)\Phi(n/x)$ in terms of multiplicative characters converts the problem of understanding $E(x,y;\alpha)$ to one of understanding $\sum_{n\in\mathcal{S}(y)}\chi(n)n^{-it}\Phi(n/x)$ for suitable Dirichlet characters χ and suitable real numbers t. We establish, on GRH, that such sums are small unless χ happens to be the principal character, and $|t|$ is small. The key step in achieving this is to bound partial Euler products $L(s,\chi;y) = \prod_{p\le y}(1-\chi(p)/p^s)^{-1}$ on GRH. The bounds for these partial Euler products that we establish are analogous to the Lindelöf bounds for Dirichlet L-functions, and the (familiar) argument is described in §5. In this fashion, we are able to understand conditionally the weighted exponential sum $E(x,y;\alpha)$, and in §6 we establish the following Theorem.

Theorem 2.4. *Assume the truth of the GRH. Let $\delta > 0$ be any fixed real number. Let x and y be large with $(\log x)^{2+\delta} \le y \le \exp((\log x)^{\frac{1}{2}-\delta})$, and let κ be defined by $y = (\log x)^\kappa$. Let $\alpha \in [0,1]$ be a real number with $\alpha = a/q+\gamma$ where $q \le \sqrt{x}$, $(a,q)=1$, and $|\gamma| \le 1/(q\sqrt{x})$.*

(1) If $|\gamma| \ge x^{\delta-1}$ then we have, for any fixed $\epsilon > 0$,

$$E(x,y;\alpha) \ll x^{\frac{3}{4}+\epsilon}.$$

(2) If $|\gamma| \le x^{\delta-1}$ then we have, writing $q = q_0 q_1$ with $q_0 \in \mathcal{S}(y)$ and all prime factors of q_1 being bigger than y, and writing $c_0 = 1 - 1/\kappa$, for any fixed $\epsilon > 0$,

$$\begin{aligned} E(x,y;\alpha) &= \frac{\mu(q_1)}{\phi(q_1)} \frac{1}{q_0^{c_0}} \prod_{p|q_0} \Big(1 - \frac{p^{c_0}-1}{p-1}\Big) \Big(c_0 \int_0^\infty \Phi(w) e(\gamma x w) w^{c_0 - 1} dw \Big) \Psi(x,y) \\ & \quad + O_\epsilon(x^{\frac{3}{4}+\epsilon}) + O_\epsilon\Big(\frac{\Psi(x,y) q_0^{-c_0+\epsilon} q_1^{-1+\epsilon}}{(1+|\gamma|x)^2} \frac{(\log\log y)}{\log y} \Big). \end{aligned}$$

The proof supposes $y \ge (\log x)^{2+\delta}$, but the result only gives a nontrivial estimate for somewhat larger y because for $\kappa \le 4$ one has the trivial estimate

$$|E(x,y;\alpha)| \ll \Psi(x,y) \ll x^{\frac{3}{4}+\epsilon}.$$

Note that by Dirichlet's theorem on Diophantine approximation one can always find $q \le \sqrt{x}$, and $(a,q) = 1$ with $|\alpha - a/q| \le 1/(q\sqrt{x})$. Theorem 2.4 then shows that $E(x,y;\alpha)$ is small unless q is small and $|\gamma|$ is small. In other words, Theorem 2.4 can be used to estimate $E(x,y;\alpha)$ on the *minor arcs* where α is not near a rational number with small denominator, and it also furnishes an asymptotic formula for our exponential sum when α lies on a *major arc*. We shall define the major and minor arcs more precisely in §7, where we use the results leading to Theorem 2.4 to complete the proof of Theorem 2.1.

We should point out that the exponential sum $\sum_{n \le x, n \in \mathcal{S}(y)} e(n\alpha)$ has been studied unconditionally be several authors, see de la Bretéche ([5], [6]), de la Bretéche and Tenenbaum ([8], [9], [10]), and de la Bretéche and Granville [7]. Our work gives better estimates, and holds in wider ranges of y, but on the other hand it relies on the truth of the GRH.

In the range of interest to us, namely y being a power of $\log x$, it is a delicate problem even to count the number of y-smooth integers up to x. One important ingredient in our work is the saddle-point method developed by Hildebrand and Tenenbaum [21] which provides an asymptotic formula for $\Psi(x,y)$ in such ranges. In §4, we survey briefly results on $\Psi(x,y)$ and extract the key results from the Hildebrand-Tenenbaum approach that we require.

Finally, in §8 we give a sieve argument that allows us to pass from all the solutions counted in Theorem 2.1 to only the primitive solutions counted in Theorem 2.2.

3. Multiplicative Character Decomposition

In this section we show how to express $e(n\alpha)\Phi(n/x)$ for $\alpha \in [0,1]$ in terms of sums over multiplicative Dirichlet characters to a certain modulus and integrals of n^{it} over t in a certain range. To achieve this we write $\alpha = a/q + \gamma$ with $(a,q) = 1$, and then our decomposition will involve Dirichlet characters mod$q$ and functions n^{it} where t is roughly of size $1 + |\gamma|x$. When $\alpha = a/q$

is a rational number, this is the familiar technique of expressing additive characters in terms of multiplicative characters, and our decomposition may be viewed as an extension of that method.

Let us first recall the decomposition of the additive character $e(an/q)$ in terms of multiplicative characters. For a Dirichlet character $\chi \pmod q$, not necessarily primitive, recall that the Gauss sum is defined by $\tau(\chi) = \sum_{b \pmod q} \chi(b) e(b/q)$.

Lemma 3.1. *Let a/q be a rational number with $(a,q)=1$.*

(1) Let n be an integer, and suppose that $(n,q)=d$. Then with $n = md$ we have

$$e\Big(\frac{an}{q}\Big) = e\Big(\frac{ma}{q/d}\Big) = \frac{1}{\phi(q/d)} \sum_{\chi \pmod{q/d}} \tau(\bar{\chi}) \chi(ma). \tag{3.1}$$

(2) One has

$$\frac{1}{\phi(q/d)^2} \sum_{\chi \pmod{q/d}} |\tau(\chi)|^2 = 1. \tag{3.2}$$

Proof. Both relations follow readily from the definition of the Gauss sum and the orthogonality relations for the Dirichlet characters $\pmod{q/d}$. □

Lemma 3.2. *(Gauss sum estimate) If $\chi \pmod q$ is primitive then $|\tau(\chi)| = \sqrt{q}$. If χ is induced by the primitive character $\chi' \pmod{q'}$ then*

$$\tau(\chi) = \mu\Big(\frac{q}{q'}\Big) \chi'\Big(\frac{q}{q'}\Big) \tau(\chi'), \tag{3.3}$$

where $\mu(n)$ is the Möbius function, and so in this case $|\tau(\chi)| \le \sqrt{q'} \le \sqrt{q}$.

Proof. This is standard; see, for example Lemma 4.1 of Granville and Soundararajan [14]. □

Now we turn to $e(n\gamma)\Phi(n/x)$ which we would like to express as an integral involving the multiplicative functions n^{it}. To do this, we define

$$\check{\Phi}(s,\lambda) := \int_0^\infty \Phi(w) e(\lambda w) w^{s-1} dw. \tag{3.4}$$

Since Φ has compact support inside $(0,\infty)$ the integral above makes sense for all complex numbers λ and s, but we shall be only interested in the case λ real. Note that $e(\lambda w)$ has the structure of an additive character while w^s has the structure of a multiplicative character so that the transform $\check{\Phi}(s,\lambda)$ plays a role analogous to the Gauss sum.

We begin by showing that $\check{\Phi}(s,\lambda)$ is small unless $1+|\lambda|$ and $1+|s|$ are of roughly the same size.

Lemma 3.3. *Let Φ be a smooth function, compactly supported in $(0,\infty)$. Let λ be real and suppose $Re(s) \ge 1/4$. Then for any non-negative integer k we have*

$$|\check{\Phi}(s,\lambda)| \ll_{k,\Phi} \min\Big(\Big(\frac{1+|\lambda|}{|s|}\Big)^k, \Big(\frac{1+|s|}{|\lambda|}\Big)^k\Big). \tag{3.5}$$

Proof. We integrate by parts k times, and can do this in two ways either using the pair of functions $\Phi(w)e(\lambda w)$ and w^{s-1}, or using the pair of functions $\Phi(w)w^{s-1}$ and $e(\lambda w)$. Integrating by parts k times using the first pair we obtain

$$\check{\Phi}(s,\lambda) = (-1)^k \int_0^\infty \frac{d^k}{dw^k}\Big(\Phi(w)e(\lambda w)\Big)\frac{w^{s+k-1}}{s(s+1)\cdots(s+k-1)}dw.$$

Since

$$\frac{d^k}{dw^k}\Big(\Phi(w)e(\lambda w)\Big) = \sum_{j=0}^k \binom{k}{j}\Phi^{(j)}(w)(2\pi i\lambda)^{k-j}e(\lambda w) \ll 2^k\sum_{j=0}^k |\Phi^{(j)}(w)|(2\pi|\lambda|)^{k-j},$$

we conclude that

$$\check{\Phi}(s,\lambda) \ll_k \frac{1}{|s|^k}\sum_{j=0}^k |\lambda|^{k-j}\int_0^\infty |\Phi^{(j)}(w)w^{s+k-1}dw| \ll_{k,\Phi} \Big(\frac{1+|\lambda|}{|s|}\Big)^k.$$

On the other hand, integrating by parts using the second pair we obtain

$$\check{\Phi}(s,\lambda) = (-1)^k \int_0^\infty \frac{d^k}{dw^k}\Big(\Phi(w)w^{s-1}\Big)\frac{e(\lambda w)}{(2\pi i\lambda)^k}dw.$$

Since

$$\begin{aligned}\frac{d^k}{dw^k}\Big(\Phi(w)w^{s-1}\Big) &= \sum_{j=0}^k \binom{k}{j}\Phi^{(j)}(w)(s-1)\cdot(s-2)\cdots(s-(k-j))w^{s-1-(k-j)}\\ &\ll_k \sum_{j=0}^k |\Phi^{(j)}(w)||s|^{k-j}|w|^{s-1-(k-j)},\end{aligned}$$

we conclude that

$$\check{\Phi}(s,\lambda) \ll_k \frac{1}{|\lambda|^k}\sum_{j=0}^k |s|^{k-j}\int_0^\infty |\Phi^{(j)}(w)w^{s-1-(k-j)}dw| \ll_{k,\Phi} \Big(\frac{1+|s|}{|\lambda|}\Big)^k.$$

□

Now we prove an analog of Lemma 3.1 for $e(n\gamma)\Phi(n/x)$.

Lemma 3.4. *Let Φ be a smooth function compactly supported in $(0,\infty)$.*

(1) For $n\in\mathbb{Z}$, we have for any positive $c=\mathrm{Re}(s)$,

$$e(n\gamma)\Phi\Big(\frac{n}{x}\Big) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\check{\Phi}(s,\gamma x)\Big(\frac{x}{n}\Big)^s ds. \tag{3.6}$$

(2) Furthermore

$$\frac{1}{2\pi}\int_{-\infty}^\infty |\check{\Phi}(c+it,\gamma x)|^2 dt = \int_{-\infty}^\infty |\Phi(e^u)e(\gamma x e^u)e^{cu}|^2 du. \tag{3.7}$$

Proof. From the definition of $\check{\Phi}$ and Mellin inversion, we obtain for $w > 0$,

$$e(\lambda w)\Phi(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \check{\Phi}(s,\lambda) w^{-s} ds.$$

We obtain (3.6) on taking $w = \frac{n}{x}$, and $\lambda = \gamma x$.

Take $s = c + it$ in the definition of $\check{\Phi}$, and change variables $w = e^u$. Thus

$$\check{\Phi}(c+it,\lambda) = \int_0^\infty \Phi(w)e(\lambda w)w^{c+it}\frac{dw}{w} = \int_{-\infty}^\infty \Phi(e^u)e(\lambda e^u)e^{cu+itu}du,$$

and we recognize that $\check{\Phi}(c+it,\lambda)$, viewed as a function of t with c and λ fixed, is the Fourier transform of $\Phi(e^u)e(\lambda e^u)e^{cu}$. Now Plancherel's theorem gives

$$\frac{1}{2\pi}\int_{-\infty}^\infty |\check{\Phi}(c+it,\lambda)|^2 ct = \int_{-\infty}^\infty |\Phi(e^u)e(\lambda e^u)e^{cu}|^2 du.$$

which, with $\lambda = \gamma x$, yields (3.7). □

Using the method of stationary phase, we can show that $|\check{\Phi}(c+it,\lambda)| \ll (1+|\lambda|)^{-\frac{1}{2}}$ and this bound is an analog of the bound $|\tau(\chi)| \le \sqrt{q}$ for Gauss sums. In our applications an L^1 version of this bound is sufficient, and we next derive such a bound from the L^2 estimate above.

Lemma 3.5. *Let λ be real and suppose that $c \ge \frac{1}{4}$. For any $\delta \ge 0$ and any $\epsilon > 0$, we have*

$$\int_{-\infty}^\infty |\check{\Phi}(c+it,\lambda)|(1+|t|)^\delta dt \ll_{\Phi,c,\epsilon} (1+|\lambda|)^{\frac{1}{2}+\delta+\epsilon}. \tag{3.8}$$

Proof. Let $\epsilon > 0$ be given. Consider first the range when $|t| > (1+|\lambda|)^{1+\epsilon}$. Using Lemma 3.3 we find that for any integer $k \ge 2$

$$\begin{aligned}\int_{|t|>(1+|\lambda|)^{1+\epsilon}} |\check{\Phi}(c+it,\lambda)|(1+|t|)^\delta dt &\ll_{k,\Phi} \int_{|t|>(1+|\lambda|)^{1+\epsilon}} \Big(\frac{1+|\lambda|}{1+|t|}\Big)^k (1+|t|)^\delta dt \\ &\ll_{k,\Phi} (1+|\lambda|)^{k-(k-\delta+1)(1+\epsilon)}.\end{aligned}$$

Choosing k suitably large, this contribution is $\ll_{\Phi,\epsilon} 1$.

Now consider the range $|t| \le (1+|\lambda|)^{1+\epsilon}$. Note that

$$\int_{|t|\le(1+|\lambda|)^{1+\epsilon}} |\check{\Phi}(c+it,\lambda)|(1+|t|)^\delta dt \ll (1+|\lambda|)^{\delta(1+\epsilon)} \int_{|t|\le(1+\lambda)^{1+\epsilon}} |\check{\Phi}(c+it,\lambda)|dt,$$

and using Cauchy-Schwarz we see that

$$\begin{aligned}\int_{|t|\le(1+\lambda)^{1+\epsilon}} |\check{\Phi}(c+it,\lambda)|dt &\le \Big(\int_{|t|\le(1+|\lambda|)^{1+\epsilon}} 1\; dt\Big)^{\frac{1}{2}} \Big(\int_{|t|\le(1+|\lambda|)^{1+\epsilon}} |\check{\Phi}(c+it,\lambda)|^2 dt\Big)^{\frac{1}{2}} \\ &\le (1+|\lambda|)^{\frac{1}{2}+\frac{1}{2}\epsilon}\Big(\int_{-\infty}^\infty |\check{\Phi}(c+it,\lambda)|^2 dt\Big)^{\frac{1}{2}} \\ &\ll_{\Phi,c} (1+|\lambda|)^{\frac{1}{2}+\frac{1}{2}\epsilon},\end{aligned}$$

upon using the Plancherel formula from Lemma 3.4(2). The Lemma follows. $\square$

Combining the formulas (3.1) and (3.6) for $\alpha = \frac{a}{q} + \gamma$, for $n \geq 1$ with $(n,q) = d$ we obtain

(3.9)
$$e(n\alpha)\Phi(\frac{n}{x}) = \Big(\frac{1}{\phi(q/d)} \sum_{\chi \pmod{\frac{q}{d}}} \tau(\bar{\chi})\chi\Big(\frac{na}{d}\Big)\Big)\Big(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \check{\Phi}(s,\gamma x)\Big(\frac{x}{n}\Big)^s dx\Big).$$

Lemma 3.1 and Lemma 3.4 exhibit parallels between the Dirichlet characters $\chi(n) \pmod q$ (the q-aspect) and the continuous family of characters $\chi_t(n) = n^{it}$ (the t-aspect). Part (1) of each lemma expresses the (weighted) additive character in terms of multiplicative characters. Gauss sums appear explicitly in Lemma 3.1, while in Lemma 3.4 the function $\check{\Phi}(c+it, \lambda)$ plays a role analogous to a Gauss sum, as it is a weighted convolution of an additive quasicharacter specified by the parameter λ against a multiplicative quasicharacter by $\chi_{c+it}(n) = n^{c+it}$. The weight function $\Phi(x)$ limits the range sampled, and Lemma 3.3 gives bounds on the size of this function. Part (2) of each lemma expresses an L^2-orthogonality relation. These orthogonality relations imply that the change of basis to multiplicative characters loses essentially nothing in the L^2-sense. However in our application, the L^1-norm is more relevant, and there is a loss in moving from additive to multiplicative characters. This is quantified in the square root losses in the both q and t aspects paralleled in the "Gauss sum" type estimates in Lemma 3.2 and Lemma 3.5, respectively.

Remark 3.6. In Theorem 2.1 we would like to subsititute the sharp cutoff weight function $\Phi(x) = \chi_{[0,1]}(x)$, but it is neither compactly supported nor continuous on $\mathbb{R}_{>0}$, and we only obtain a lower bound (2.2) rather than the expected asymptotic formula. Here we note in passing that the transform $\check{\Phi}(s,\lambda)$ given in (3.4) is an interesting special function. Namely, for $Re(\lambda) < 0$, we have

$$\check{\Phi}(s,\lambda) = \int_0^1 e^{\lambda x} x^{s-1} dx = (-\lambda)^{-s}\gamma(s, -\lambda), \tag{3.10}$$

where $\gamma(s,z) = \int_0^z e^{-u}u^{s-1}du$ is the incomplete gamma function. The incomplete gamma function is related to Kummer's confluent hypergeometric function

$$M(a,b,z) := {}_1F_1(a,b;z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!} + \cdots,$$

by special function formulas (see Chapter 13 of [1]) which yield

$$s(-\lambda)^{-s}\gamma(s,-\lambda) = M(s, s+1, \lambda) = e^{\lambda}M(1, s+1, \lambda). \tag{3.11}$$

The last equality is a special case of Kummer's transformation $M(a,b,z) = e^z M(b-a, b, z)$. The known analytic properties of the function $M(a,b,z)$

(in three complex variables) give an analytic continuation of $\frac{1}{\Gamma(s)}\check{\Phi}(s,\lambda)$ to an entire function of two complex variables. It follows that $\check{\Phi}(s,\lambda)$ has no singularities in the λ-variable, but for generic λ it has simple poles in the s-variable at the nonpositive integers.

4. A brief survey of results on $\Psi(x,y)$

In this section we collect together several results on estimates for $\Psi(x,y)$. A comprehensive survey of this topic is given by Hildebrand and Tenenbaum [22], and we give here a very brief description of the salient points.

When y is not too small in relation to x, then on writing $y = x^{\frac{1}{u}}$, we have that $\Psi(x,y) \sim x\rho(u)$ where ρ is the Dickman function which is defined by $\rho(u) = 1$ for $0 \le u \le 1$, and for $u \ge 1$ is defined by the differential-difference equation $u\rho'(u) = -\rho(u-1)$. The most precise version of this result is due to Hildebrand [20] who showed that for all large x and $y \ge \exp((\log\log x)^{5/3+\epsilon})$ we have

$$\Psi(x,y) = x\rho(u)\Big(1 + O_\epsilon\Big(\frac{u\log(u+1)}{\log x}\Big)\Big). \tag{4.1}$$

Here we are particularly interested in the range when y is a power of $\log x$. This is the relevant range for our main results, but it lies outside the range covered by Hildebrand's (4.1). Indeed in this range, the behavior of $\Psi(x,y)$ is known to be sensitive to the fine distribution of primes and location of the zeros of $\zeta(s)$. In 1984 Hildebrand [18] showed that the Riemann hypothesis is equivalent to the assertion that for each $\epsilon > 0$ and $1 \le u \le y^{1/2-\epsilon}$ there is a uniform estimate

$$\Psi(x,y) = x\rho(u)\exp(O_\epsilon(y^\epsilon)). \tag{4.2}$$

Moreover, assuming the Riemann hypothesis, he showed that for each $\epsilon > 0$ and $1 \le u \le y^{1/2-\epsilon}$ the stronger uniform estimate

$$\Psi(x,y) = x\rho(u)\exp\Big(O_\epsilon\Big(\frac{\log(u+1)}{\log y}\Big)\Big) \tag{4.3}$$

holds. On choosing $y = (\log x)^\alpha$ for $\alpha > 2$, this latter estimate yields

$$\Psi(x,(\log x)^\alpha) \asymp x\rho(u), \tag{4.4}$$

which provides only an order of magnitude estimate for the size of $\Psi(x,y)$. Furthermore if the Riemann hypothesis is false then $\Psi(x,y)$ must sometimes exhibit large oscillations away from the value $x\rho(u)$ for some (x,y) in these ranges. In 1986 Hildebrand [19] obtained further results indicating that when $y < (\log x)^{2-\epsilon}$ one should not expect any smooth asymptotic formula for $\Psi(x,y)$ in terms of the y-variable to hold.

Since we assume GRH in this paper, we may access these conditional results of Hildebrand. However a less explicit asymptotic formula for $\Psi(x,y)$ developed by Hildebrand and Tenenbaum [21] is more useful for us. Before discussing the results from their saddle point method, we note a useful, and uniform, elementary asymptotic for $\log \Psi(x,y)$; see Theorem 1.4 of [22]. Uniformly for all $x \geq y \geq 2$ there holds

$$\log \Psi(x,y) = \Big(\frac{\log x}{\log y} \log\Big(1+\frac{y}{\log x}\Big)+\frac{y}{\log y}\log\Big(1+\frac{\log x}{y}\Big)\Big)\Big(1+O\Big(\frac{1}{\log y}+\frac{1}{\log\log x}\Big)\Big).$$

If $y=(\log x)^{\alpha}$, with $\alpha \geq 1$ then it follows that

$$\Psi(x,y) = x^{1-\frac{1}{\alpha}} \exp\Big(O\Big(\frac{\log x}{\log\log x}\Big)\Big). \tag{4.5}$$

We define

$$\zeta(s;y) := \sum_{n\in\mathcal{S}(y)} n^{-s} = \prod_{p\leq y}\Big(1-\frac{1}{p^s}\Big)^{-1},$$

and by Perron's formula we may write, for any $c>0$,

$$\Psi(x,y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s;y)x^s\frac{ds}{s}. \tag{4.6}$$

The method of Hildebrand and Tenenbaum makes a careful choice for the line of integration (c). Precisely, they choose c such that the quantity $x^{\sigma}\zeta(\sigma;y)$ is minimized over all $0<\sigma\leq\infty$. With a little calculus, this quantity is minimized when $c=c(x,y)$ is the unique solution to

$$-\phi_1(c;y) := -\frac{d}{dc}\log\zeta(c;y) = \sum_{p\leq y}\frac{\log p}{p^c-1} = \log x, \tag{4.7}$$

where $\phi_j(c;y)$ denotes the j-th derivative with respect to s of $\log\zeta(s;y)$. The quantity[1] $c(x,y)$ is a saddle-point for the function $x^s\zeta(s;y)$ in the sense that $|x^s\zeta(s;y)|$ is minimized over real values of $s\in(0,\infty)$, but is maximized over values $s=c+it$ for $t\in\mathbb{R}$. With this choice for the line of integration, Hildebrand and Tenenbaum found that the integral in (4.6) is dominated by the portion of the integral near the real axis, and were able to evaluate this contribution. We now quote their result, see Theorem 1 of [21].

Theorem 4.1. (Hildebrand-Tenenbaum) *We have uniformly for $x\geq y\geq 2$,*

$$\Psi(x,y) = \frac{x^c\zeta(c;y)}{c\sqrt{2\pi\phi_2(c,y)}}\left(1+O\left(\frac{1}{u}+\frac{\log y}{y}\right)\right), \tag{4.8}$$

in which $c=c(x,y)$, and $y=x^{\frac{1}{u}}$.

The following result, Theorem 2 of [21], concerns the size of $c(x,y)$ and of the denominator in (4.8), involving

$$\phi_2(c;y) = \frac{d^2}{dc^2}\log\zeta(c;y) = \sum_{p\leq y}\frac{p^c(\log p)^2}{(p^c-1)^2}.$$

[1]Hildebrand and Tenenbaum denote this quantity $\alpha(x,y)$ and abbreviate it to α.

Theorem 4.2. (Hildebrand-Tenenbaum) *We have uniformly for $x \geq y \geq 2$,*

$$c(x,y) = \frac{\log\left(1+\frac{y}{\log x}\right)}{\log y}\left(1+O\left(\frac{\log\log(1+y)}{\log y}\right)\right), \tag{4.9}$$

and

$$\phi_2(c(x,y),y) = \left(1+\frac{\log x}{y}\right)\log x \cdot \log y\left(1+O\left(\frac{1}{\log(1+u)}+\frac{1}{\log y}\right)\right). \tag{4.10}$$

An immediate consequence of (4.9) is that for fixed $\delta > 0$, and $y = (\log x)^\kappa$ with $\kappa \geq 1+\delta$ we have

$$c(x,y) = 1 - \frac{1}{\kappa} + O_\delta\Big(\frac{\log\log y}{\log y}\Big). \tag{4.11}$$

While the asymptotic in Theorem 4.1 may be a little difficult to parse, it provides an elegant and useful means of obtaining the "local behavior" of $\Psi(x,y)$, given as follows – see Theorem 3 of [21].

Theorem 4.3. (Hildebrand-Tenenbaum) *We have uniformly for $x \geq y \geq 2$ and $1 \leq k \leq y$,*

$$\Psi(kx,y) = \Psi(x,y)k^{c(x,y)}\left(1+O\left(\frac{\log y}{\log x}+\frac{\log y}{y}\right)\right). \tag{4.12}$$

This result can be used to show that the behavior of $\Psi(x,y)$ with $y = (\log x)^\kappa$ changes qualitatively at $\kappa = 1$, having a "phase transition" there. As $x \to \infty$, Theorem 4.3 implies that when $\kappa \leq 1$ one has

$$\Psi(kx,y) = (1+o(1))\Psi(x,y),$$

whereas for $\kappa > 1$ one has

$$\Psi(kx,y) = \left(k^{1-\frac{1}{\kappa}} + o(1)\right)\Psi(x,y).$$

For later use, we state three estimates of Hildebrand and Tenenbaum (restricted to the range $y \geq \log x$) as lemmas.

Lemma 4.4. (Hildebrand and Tenenbaum) *Let x and y be large with $y \geq \log x$, and let $s = c + i\tau$ with $c = c(x,y)$ and real τ. Uniformly in the region $1/\log y \leq |\tau| \leq y$ we have*

$$\Big|\frac{\zeta(s;y)}{\zeta(c;y)}\Big| \ll \exp\Big(-c_0\frac{u\tau^2}{(1-c)^2+\tau^2}\Big). \tag{4.13}$$

Proof. This is a special case of Lemma 8 of [21]. □

Lemma 4.5. (Hildebrand and Tenenbaum) *Let $0<\beta<1$ be fixed. Then uniformly for $x \ge y \ge 2$,*

$$\Psi(x,y) = \frac{1}{2\pi i}\int_{c-\frac{i}{\log y}}^{c+\frac{i}{\log y}} \zeta(s;y)\frac{x^s}{s}ds$$
$$+O_\beta\Big(x^c\zeta(c,y)\Big(\exp(-(\log y)^{\frac{3}{2}-\beta}) + \exp\Big(-c_6\frac{u}{(\log 2u)^2}\Big)\Big)\Big), \tag{4.14}$$

with $c=c(x,y)$, and $c_6>0$ an absolute constant.

Proof. This is Lemma 10 of [21]. □

Lemma 4.6. (Hildebrand and Tenenbaum) *If x and y are large, and $y \ge \log x$,*

$$\frac{1}{2\pi i}\int_{c-\frac{i}{\log y}}^{c+\frac{i}{\log y}} \zeta(s;y)\frac{x^s}{s}ds = \frac{x^c\zeta(c;y)}{c\sqrt{2\pi\phi_2(c;y)}}\left(1+O\left(\frac{1}{u}\right)\right), \tag{4.15}$$

with $c=c(x,y)$. Moreover, the same estimate holds for

$$\frac{1}{2\pi}\int_{c-\frac{i}{\log y}}^{c+i\frac{i}{\log y}} |\zeta(s;y)\frac{x^s}{s}||ds| = \frac{x^c\zeta(c;y)}{c\sqrt{2\pi\phi_2(c;y)}}\left(1+O\left(\frac{1}{u}\right)\right). \tag{4.16}$$

Proof. This is Lemma 11 of [21], restricted to the range $y \ge \log x$. □

The agreement in size of the integral (4.15) with the absolute value estimate (4.16), is a key feature of the integral being at the saddle point. We remark that Lemma 4.5 and Lemma 4.6 are major ingredients used by Hildebrand and Tenenbaum in proving Theorem 4.1.

5. Bounds for partial L-functions on GRH

It is well-known that the generalized Riemann hypothesis implies the generalized Lindelöf hypothesis: If $\chi \pmod q$ is a primitive character and s is a complex number with $\mathrm{Re}(s) \ge 1/2$, then for any $\epsilon>0$ we have $|L(s,\chi)| \ll_\epsilon (q|s|)^\epsilon$. Our aim in this section is to establish a corresponding conditional estimate for the partial Euler products

$$L(s,\chi;y) := \prod_{p\le y}\Big(1-\chi(p)p^{-s}\Big)^{-1}.$$

Proposition 5.1. *Assume the truth of the GRH. Let $\chi \pmod q$ be a primitive Dirichlet character. For any $\epsilon>0$, and s a complex number with $Re(s)=\sigma \ge 1/2+\epsilon$, we have*

$$|L(s,\chi;y)| \ll_\epsilon (q|s|)^\epsilon. \tag{5.1}$$

For the trivial character we have, with $\sigma=\mathrm{Re}(s)\ge 1/2+\epsilon$,

$$|\zeta(s;y)| \ll_\epsilon \exp\Big(\frac{y^{1-\sigma}}{(1+|t|)\log y}\Big)|s|^\epsilon. \tag{5.2}$$

We shall prove Proposition 5.1 by developing conditional estimates for $\sum_{n\le u}\Lambda(n)\chi(n)n^{-it}$. These estimates follow from standard "explicit formula" arguments connecting such prime sums with zeros of the corresponding L-function, and we shall be brief in sketching their proofs.

Lemma 5.2. *Let $\chi \pmod q$ be a primitive Dirichlet character, and let t be a real number. Let $\rho=\beta+i\gamma$ denote a typical zero of the Dirichlet L-function $L(s,\chi)$. Let $\delta(\chi)=1$ if $q=1$ and χ is the principal character, and $\delta(\chi)=0$ otherwise. Then for $u\ge 2$ and any parameter $T\ge 2$ we have*

$$\begin{aligned}\sum_{n\le u}\Lambda(n)\chi(n)n^{-it} &= \delta(\chi)\frac{u^{1-it}}{1-it}-\sum_{\substack{0<\beta<1\\ |\gamma-t|\le T}}\frac{u^{\rho-it}}{\rho-it}\\ &\quad+O\Big(\Big(1+\frac{u}{T}\Big)(\log(qu(T+|t|)))^2+\sum_{|\rho|\le 1}\frac{1}{|\rho|}\Big).\end{aligned}$$

Proof. This unconditional result may be derived by following the method given in Chapters 17 and 19 of Davenport [11]. We start with Perron's formula

(5.3)
$$\frac{1}{2\pi i}\int_{1+1/\log u-i\infty}^{1+1/\log u+i\infty}-\frac{L'}{L}(w+it,\chi)\frac{u^w}{w}dw=\sum_{n\le u}\Lambda(n)\chi(n)n^{-it}+O(\log u),$$

Now for each $T\ge 2$ we may find T_1 and T_2 with $|T_1+T|\le 1$ and $|T_2-T|\le 1$ such that $|L'/L(c+iT_j+it)|\ll(\log(q(T+|t|)))^2$ for all $-\frac12\le c\le 1+1/\log x$. We truncate the integral in (5.3) to the line segment $[1+1/\log u+iT_1, 1+1/\log u+iT_2]$ and incur an error of $O(u(\log u)^2/T)$. We now shift the line of integration to $\mathrm{Re}(w)=-\frac12$, using a rectangular contour. In view of our choice for the heights T_1 and T_2, the horizontal sides contribute $O(u(\log(q(T+|t|))^2/T)$. The vertical side of the box with $\mathrm{Re}(w)=-\frac12$ contributes $O((\log qu(T+|t|))^2/\sqrt{u})$, upon using the functional equation to estimate L'/L on this line. The net contribution of the error terms discussed so far is

$$\ll\Big(\frac{u}{T}+1\Big)(\log(qu(T+|t|)))^2.$$

It remains lastly to discuss the residues of the poles encountered while shifting our contour. If $q=1$ and χ is the principal character, there is a pole at $w=1-it$ which leaves the residue $u^{1-it}/(1-it)$. If ρ is a zero of $L(s,\chi)$ with $0<\beta<1$ and $T_1\le\gamma-t\le T_2$ then there is a pole at $w=\rho-it$ in our contour shift. The contribution of these poles is

$$-\sum_{\substack{0<\beta<1\\ T_1<\gamma-t\le T_2}}\frac{u^{\rho-it}}{\rho-it}=-\sum_{\substack{0<\beta<1\\ |\gamma-t|\le T}}\frac{u^{\rho-it}}{\rho-it}+O\Big(\frac{u}{T}\log(q(T+|t|))\Big),$$

since the conditions $T_1<\gamma-t<T_2$ and $|\gamma-t|\le T$ are different for at most $\ll\log(q(T+|t|))$ zeros. Finally there is a pole at $w=0$ and, if $\chi(-1)=1$,

$q>1$ and $-t\in[T_1,T_2]$ a pole at $w=-it$. The residues at these poles may be treated as in Chapter 19 of Davenport [11] and they contribute an amount $\ll \log(qu(T+|t|))+\sum_{|\rho|\le 1}1/|\rho|$. This sum over $|\rho|\le 1$ is to account for the case where there is a Siegel zero very near 1 (and hence a corresponding zero very near 0).

Assembling these observations together, we obtain the Lemma. $\square$

Lemma 5.3. *Assume the truth of the GRH. If $\chi \pmod q$ is a primitive Dirichlet character with $q>1$, then for $u\ge 1$ and all real t we have*

$$\sum_{n\le u}\Lambda(n)\chi(n)n^{-it}\ll \sqrt{u}(\log u)\log(qu(|t|+2)). \tag{5.4}$$

In the case of the principal character (and so $q=1$), we have for $u\ge 1$ and all real t,

$$\sum_{n\le u}\Lambda(n)n^{-it}=\frac{u^{1-it}}{1-it}+O(\sqrt{u}(\log u)\log(u(|t|+2))). \tag{5.5}$$

Proof. We apply Lemma 5.2 choosing $T=u^2$. We shall use GRH to bound the sums over zeros appearing there, and recall that there are $\ll (\log(q(2+|z|))$ zeros of $L(s,\chi)$ in $|\gamma-z|\le 1$. Thus we obtain that

$$\begin{aligned}\sum_{n\le u}\Lambda(n)\chi(n)n^{-it} &= \delta(\chi)\frac{u^{1-it}}{1-it}+O\Big(\sum_{|\gamma-t|\le T}\frac{\sqrt{u}}{1+|t-\gamma|}+(\log(qu(2+|t|)))^2\Big)\\ &= \delta(\chi)\frac{u^{1-it}}{1-it}+O\Big(\sqrt{u}(\log(qu(2+|t|)))\log u+(\log(qu(2+|t|)))^2\Big).\end{aligned}$$

If $\log(qu(2+|t|)\le\sqrt{u}$ then the second error term above may be absorbed into the first, and our Lemma follows. If $\log(qu(2+|t|)\ge\sqrt{u}$ then the stated estimates are weaker than the trivial bound $\sum_{n\le u}\Lambda(n)\chi(n)n^{-it}\ll u$, and so our Lemma holds in this case also.

$\square$

Proof of Proposition 5.1. From the definition of $L(s,\chi;y)$ we have that

$$|L(s,\chi;y)|=\exp\Big(\mathrm{Re}(\log L(s,\chi;y))\Big)\ll\exp\Big(\mathrm{Re}\sum_{n\le y}\frac{\Lambda(n)\chi(n)n^{-it}}{n^\sigma\log n}\Big).$$

If $(\log(q(2+|t|)))^2\ge y$, then using the prime number theorem we have that

$$\sum_{n\le y}\frac{\Lambda(n)\chi(n)n^{-it}}{n^\sigma\log n}\ll\sum_{n\le(\log(q(2+|t|)))^2}\frac{\Lambda(n)}{\sqrt{n}\log n}\ll\frac{\log(q(2+|t|))}{\log\log(q(2+|t|))}, \tag{5.6}$$

and the bounds of the Lemma hold.

Suppose now that $y\ge(\log(q(2+|t|)))^2$. We use the estimate (5.6) above for the terms $n\le(\log(q(2+|t|)))^2$, and use partial summation and Lemma

5.3 for larger values of n. Thus we find that

$$\sum_{n\le y}\frac{\Lambda(n)\chi(n)n^{-it}}{n^\sigma\log n}=O\Big(\frac{\log(q(2+|t|))}{\log\log(q(2+|t|))}\Big)+\int_{(\log q(2+|t|))^2}^{y}\frac{1}{z^\sigma\log z}d\Big(\sum_{n\le z}\Lambda(n)n^{-it}\chi(n)\Big).$$

Suppose first that $q>1$. Integrating by parts, and using (5.4) we see that the integral above is

$$\begin{aligned}&\ll\ (\log(q(2+|t|)))^{2-2\sigma}+\int_{(\log(q(2+|t|)))^2}^{y}\sqrt{z}(\log(qz(2+|t|)))\Big(\frac{\sigma}{z^{\sigma+1}}+\frac{1}{z^{\sigma+1}\log z}\Big)dz\\ &\ll\ \frac{\sigma}{\sigma-\frac12}(\log(q(2+|t|)))^{2-2\sigma}.\end{aligned}$$

If $\sigma\ge\frac12+\epsilon$ then the above estimates readily imply (5.1).

The case when $q=1$ is similar, but we appeal to (5.5) in place of (5.4) above. This leads to including an extra main term in our sum above of size $y^{1-\sigma+it}/((1-it)\log y)$, and thus we obtain (5.2).

$\square$

6. The weighted exponential sum $E(x,y;\alpha)$

Our aim in this section is to understand the weighted sum $E(x,y;\alpha)=\sum_{n\in\mathcal{S}(y)}e(n\alpha)\Phi(n/x)$. We shall use the decomposition into multiplicative characters developed in §3 together with the GRH bounds for partial L-functions developed in §5. Here Φ is treated as fixed, and all constants in O-symbols depend on it.

Proposition 6.1. *Assume the truth of the GRH. Let α be a real number in $[0,1]$ and write $\alpha=a/q+\gamma$ with $(a,q)=1$, $q\le\sqrt{x}$, and $|\gamma|\le 1/(q\sqrt{x})$. Then*

$$E(x,y;\alpha)=M(x,y;q,\gamma)+O(x^{\frac34+\epsilon}), \tag{6.1}$$

where the "local main term" $M(x,y;q,\gamma)$ is defined by

$$M(x,y;q,\gamma)=\sum_{n\in\mathcal{S}(y)}\frac{\mu(\frac{q}{(q,n)})}{\phi(\frac{q}{(q,n)})}e(n\gamma)\Phi\Big(\frac{n}{x}\Big). \tag{6.2}$$

Proof. We begin by remarking that Dirichlet's theorem on diophantine approximation guarantees the existence of decompositions $\alpha=a/q+\gamma$ with $(a,q)=1$, $q\le\sqrt{x}$ and $|\gamma|\le 1/(q\sqrt{x})$. Writing $n\in\mathcal{S}(y)$ as dm where $d=(n,q)$ we see that

$$E(x,y;\alpha)=\sum_{\substack{d|q\\ d\in\mathcal{S}(y)}}\ \sum_{\substack{m\in\mathcal{S}(y)\\ (m,\frac{q}{d})=1}}e\Big(\frac{am}{q/d}\Big)e(md\gamma)\Phi\Big(\frac{md}{x}\Big).$$

Using now Lemma 3.1 we find that
(6.3)
$$E(x,y;\alpha) = \sum_{\substack{d|q \\ d\in\mathcal{S}(y)}} \frac{1}{\phi(q/d)} \sum_{\chi (\mathrm{mod}\ q/d)} \chi(a)\tau(\bar\chi) \sum_{\substack{m\in\mathcal{S}(y)\\ (m,\frac{q}{d})=1}} e(md\gamma)\chi(m)\Phi\Big(\frac{md}{x}\Big).$$

Consider first the contribution of the principal character (mod q/d). The Gauss sum for the principal character (mod q/d) equals $\mu(q/d)$, and hence the contribution of the principal characters to (6.3) is

$$\sum_{\substack{d|q \\ d\in\mathcal{S}(y)}} \frac{\mu(q/d)}{\phi(q/d)} \sum_{\substack{m\in\mathcal{S}(y)\\ (m,\frac{q}{d})=1}} e(md\gamma)\Phi\Big(\frac{md}{x}\Big) = \sum_{n\in\mathcal{S}(y)} \frac{\mu(q/(q,n))}{\phi(q/(q,n))} e(n\gamma)\Phi\Big(\frac{n}{x}\Big) = M(x,y;q,\gamma).$$

This is the main term isolated in our Proposition, and we must show that the contribution of the non-principal characters to (6.3) is $O(x^{\frac{3}{4}+\epsilon})$.

We shall establish using Proposition 5.1 that if χ is not the principal character (mod q/d) then

$$\frac{\sqrt{q}}{\sqrt{d}}\Big|\sum_{m\in\mathcal{S}(y)} e(md\gamma)\chi(m)\Phi\Big(\frac{md}{x}\Big)\Big| \ll x^{\frac{3}{4}+\epsilon}. \tag{6.4}$$

Assuming this for the present, since $|\tau(\overline{\chi})| \le \sqrt{q/d}$ for all characters χ (mod q/d) by Lemma 3.2, we see that the contribution of the non-principal characters to (6.3) is bounded by

$$\ll \sum_{\substack{d|q \\ d\in\mathcal{S}(y)}} \frac{1}{\phi(q/d)} \sum_{\substack{\chi \pmod{q/d} \\ \chi\neq\chi_0}} x^{\frac{3}{4}+\epsilon} \ll x^{\frac{3}{4}+\epsilon} d(q) \ll x^{\frac{3}{4}+\epsilon}.$$

Thus to finish the proof of our Proposition, we need only establish (6.4). Using Lemma 3.4 we see that for any $c>0$

$$\begin{aligned}\sum_{m\in\mathcal{S}(y)} e(md\gamma)\chi(m)\Phi\Big(\frac{md}{x}\Big) &= \sum_{m\in\mathcal{S}(y)} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \check\Phi(s,\gamma x)\Big(\frac{x}{dm}\Big)^s ds \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} L(s,\chi;y)\check\Phi(s,\gamma x)\Big(\frac{x}{d}\Big)^s ds\end{aligned}$$

where the interchange of the sum and integral is justified by the absolute convergence of $L(s,\chi;y)$ for any $\mathrm{Re}(s)>0$. We now take $c = 1/2+\epsilon$ and invoke the GRH bound from Proposition 5.1 which gives $L(s,\chi;y) \ll (q|s|)^\epsilon$. Note that Proposition 5.1 applies to primitive characters χ, but we may extend it easily to imprimitive characters as follows. Suppose χ is induced from a primitive character $\tilde\chi$ (mod $\tilde q$) then we have $|L(s,\chi;y)| \le |L(s,\tilde\chi;y)|\prod_{p|(q/\tilde q)}(1+1/\sqrt{p}) \ll (q|s|)^\epsilon$ upon using the bound of Proposition

5.1 for $L(s,\tilde{\chi};y)$. It follows that

$$\Big|\sum_{m\in\mathcal{S}(y)} e(md\gamma)\chi(m)\Phi\Big(\frac{md}{x}\Big)\Big| \ll \Big(\frac{x}{d}\Big)^{\frac12+\epsilon} q^{\epsilon}\int_{-\infty}^{\infty} |\check{\Phi}(\tfrac12+\epsilon+it,\gamma x)|(1+|t|)^{\epsilon}dt.$$

Using Lemma 3.5, we conclude that

$$\frac{\sqrt{q}}{\sqrt{d}}\Big|\sum_{m\in\mathcal{S}(y)} e(md\gamma)\chi(m)\Phi\Big(\frac{md}{x}\Big)\Big| \ll (\frac{1}{d})^{1+\epsilon}x^{\frac12+\epsilon}q^{\frac12+\epsilon}(1+|\gamma|x)^{\frac12+\epsilon} \ll x^{\frac34+\epsilon},$$

since $q\le\sqrt{x}$ and $q|\gamma x|\le\sqrt{x}$. This establishes (6.4) and hence our Proposition.

□

We now consider the "local main terms" $M(x,y;q,\gamma)$, and start with a simple reduction.

Lemma 6.2. *Given a positive integer q, write $q=q_0q_1$, in which $q_0\in\mathcal{S}(y)$ and q_1 is divisible only by primes larger than y. Let $M(x,y;q,\gamma)$ be as in Proposition 6.1. Then*

$$M(x,y;q,\gamma)=\frac{\mu(q_1)}{\phi(q_1)}M(x,y;q_0,\gamma). \tag{6.5}$$

Proof. This is immediate from the definition (6.2). □

It remains to treat the case $q_0\in\mathcal{S}(y)$, and here we use the saddle point method of Hildebrand and Tenenbaum discussed in §4 to obtain an understanding of this main term. In the following result the lower bound $y\ge(\log x)^{2+\delta}$ is imposed only as a necessary condition for nontriviality of the estimate.

Proposition 6.3. *Assume the truth of the GRH. Let x and y be large, and assume that $(\log x)^{2+\delta}\le y\le\exp((\log x)^{\frac12-\delta})$. Let $c=c(x,y)$ denote the Hildebrand-Tenenbaum saddle point value given in section 4. Suppose $q_0\in\mathcal{S}(y)$ with $q_0<\sqrt{x}$, let γ be real with $|\gamma|\le 1/(q_0\sqrt{x})$, and let $M(x,y;q_0,\gamma)$ be as in Proposition 6.1. Then we have:*

(1) If $|\gamma|\ge x^{\delta-1}$ then, for any fixed $\epsilon>0$,

$$|M(x,y;q_0,\gamma)|\ll x^{\frac34+\epsilon}q_0^{-\frac34+\epsilon}.$$

(2) If $|\gamma|\le x^{\delta-1}$ we have , for any fixed $\epsilon>0$,

$$\begin{aligned}M(x,y;q_0,\gamma) &= \frac{1}{q_0^c}\prod_{p|q_0}\Big(1-\frac{p^c-1}{p-1}\Big)(c\check{\Phi}(c,\gamma x))\Psi(x,y)+O_\epsilon(x^{\frac34+\epsilon}q_0^{-\frac34+\epsilon})\\ &\qquad +O_\epsilon\Big(\frac{\Psi(x,y)q_0^{-c+\epsilon}}{(\log y)(1+|\gamma|x)^2}\Big).\end{aligned}$$

Proof. Using Lemma 3.4 we see that for any $\sigma > 0$ we have

$$M(x,y;q_0,\gamma) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{n\in\mathcal{S}(y)} \frac{\mu(q_0/(q_0,n))}{\phi(q_0/(q_0,n))}\frac{1}{n^s}\check{\Phi}(s,\gamma x)x^s ds.$$

We now may write

$$\sum_{n\in\mathcal{S}(y)} \frac{\mu(q_0/(q_0,n))}{\phi(q_0/(q_0,n))}\frac{1}{n^s} = \zeta(s;y)H(s;q_0),$$

where $H(s;q_0)$ is a Dirichlet series involving only integers with prime factors dividing q_0. For each prime $p|q_0$ let $\nu_p(q_0)$ denote the exact power of p dividing q_0, so that $\nu_p(q_0) \geq 1$. Then

$$\begin{aligned} H(s;q_0) &= \prod_{p|q_0}\Big(1-\frac{1}{p^s}\Big)\Big(\sum_{k=\nu_p(q_0)-1}^{\infty}\frac{\mu(p^{\nu_p(q_0)}/(p^{\nu_p(q_0)},p^k))}{\phi(p^{\nu_p(q_0)}/(p^{\nu_p(q_0)},p^k))}\frac{1}{p^{ks}}\Big) \\ &= \prod_{p|q_0}\Big(1-\frac{1}{p^s}\Big)\Big(-\frac{1}{(p-1)}\frac{1}{p^{(\nu_p(q_0)-1)s}}+\frac{1}{p^{\nu_p(q_0)s}}\Big(1-\frac{1}{p^s}\Big)^{-1}\Big) \\ &= \frac{1}{q_0^s}\prod_{p|q_0}\Big(1-\frac{p^s-1}{p-1}\Big). \end{aligned} \tag{6.6}$$

We may now write our integral formula as

$$M(x,y;q_0,\gamma) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\zeta(s;y)H(s;q_0)\check{\Phi}(s,\gamma x)x^s ds. \tag{6.7}$$

We deform the integral above, replacing it by an integral over a piecewise linear contour consisting of (i) a line segment c_1+it with t going from $-y$ to y, (ii) a horizontal line segment going from c_1+iy to $1/2+\epsilon+iy$ and another going from $1/2+\epsilon-iy$ to c_1-iy, and (iii) a vertical line segment going from $1/2+\epsilon+iy$ to $1/2+\epsilon+i\infty$ and another going from $1/2+\epsilon-i\infty$ to $1/2+\epsilon-iy$. The shift of contour is permitted because the integrand is holomorphic and bounded in vertical strips $0<\sigma_1<\mathrm{Re}(s)<\sigma_2$ and is rapidly decreasing as $|\mathrm{Im}(s)|\to\infty$ using the bound of Lemma 3.3. The proofs of (1) and (2) will choose different values of c_1. In the calculations below it will be useful to keep in mind that for all s with $1/2\leq\mathrm{Re}(s)\leq 1$ we have

$$|H(s;q_0)| \leq d(q_0)q_0^{-\mathrm{Re}(s)} \ll q_0^{-\mathrm{Re}(s)+\epsilon}.$$

We consider first the vertical line segments given in case (iii) above, which do not depend on the choice of c_1. Using Proposition 5.1 (which assumes GRH), we see that the contribution of these segments to $M(x,y;q_0,\gamma)$ is

$$\begin{aligned} &\ll q_0^{-\frac{1}{2}+\epsilon}x^{\frac{1}{2}+\epsilon}\int_{-\infty}^{\infty}(1+|t|)^{\epsilon}|\check{\Phi}(\tfrac{1}{2}+\epsilon+it,\gamma x)|dt \\ &\ll q_0^{-\frac{1}{2}+\epsilon}x^{\frac{1}{2}+\epsilon}(1+|\gamma x|)^{\frac{1}{2}+\epsilon} \ll q_0^{-\frac{3}{4}+\epsilon}x^{\frac{3}{4}+\epsilon}, \end{aligned} \tag{6.8}$$

upon using Lemma 3.5 and that $|\gamma|x \le \sqrt{x}/q_0$ To handle the remaining integrals, we distinguish two cases depending on whether $|\gamma| \ge x^{\delta-1}$ or not.

(1) First we treat the case when $|\gamma| \ge x^{\delta-1}$. In this case we will choose $c_1 = 1+\epsilon$. Taking k suitably large in Lemma 3.3 (depending on δ) we find that $\check{\Phi}(s,\gamma x) \ll x^{-1}$ for all s on the portions of the contour given in (i) and (ii) above. Consider the contribution to the integral of the horizontal line segments in (ii). Proposition 5.1 gives that

$$|\zeta(s;y)| \ll_\epsilon \exp\Big(\frac{y^{1-\sigma}}{(1+y)\log y}\Big)|s|^\epsilon \ll_\epsilon |s|^\epsilon \ll y^\epsilon, \tag{6.9}$$

and so this contribution is

$$\ll y^\epsilon x^{-1}\int_{1/2+\epsilon}^{c_1} x^\sigma q_0^{-\sigma+\epsilon} d\sigma \ll x^\epsilon.$$

Next consider the vertical line segment given in (i). Here we bound $|\zeta(s;y)|$ by $\zeta(c_1,y) \ll_\epsilon 1$, so this segment contributes

$$\ll x^{c_1} q_0^{-c+\epsilon} x^{-1}\zeta(c_1;y)y \ll_\epsilon x^\epsilon.$$

Combining these estimates with (6.8) we conclude that when $|\gamma| \ge x^{\delta-1}$ we have $M(x,y;q_0,\gamma) \ll x^{\frac{3}{4}+\epsilon} q_0^{-\frac{3}{4}+\epsilon}$, as claimed.

(2) Now we turn to the case when $|\gamma| \le x^{\delta-1}$. In this case we choose $c_1 = c$ to be the Hildebrand-Tenenbaum saddle point value. We use Lemma 3.3 with $k=2$ which gives that $\check{\Phi}(s,\gamma) \ll |s|^2/(1+|\gamma|x)^2$. Now consider the contribution to the integral of the horizontal line segments described in (ii). As in (6.9) above, Proposition 5.1 gives that $|\zeta(s;y)| \ll |s|^\epsilon \ll y^\epsilon$, and so the contribution of these line segments to the integral giving $M(x,y;q_0,\gamma)$ is

$$\ll y^\epsilon \frac{y^2}{(1+|\gamma|x)^2}\int_{1/2+\epsilon}^{c} x^\sigma q_0^{-\sigma+\epsilon} d\sigma \ll \frac{y^{2+\epsilon}}{(1+|\gamma|x)^2} q_0^{-c+\epsilon} x^c.$$

Note that, using (4.6),

$$\log\zeta(c;y) \ge \sum_{p\le y} p^{-c} \ge \frac{1}{2\log y}\sum_{p\le y}\frac{\log p}{p^c-1} = \frac{\log x}{2\log y},$$

and so the contribution of the horizontal line segments is, using Theorems 4.1 and 4.2

$$\ll \frac{y^{2+\epsilon}}{(1+|\gamma|x)^2} q_0^{-c+\epsilon} x^c \zeta(c;y)\exp\Big(-\frac{\log x}{2\log y}\Big) \ll q_0^{-c+\epsilon}\Psi(x,y)\frac{y^{2+\epsilon}\log x}{(1+|\gamma|x)^2}\exp\Big(-\frac{\log x}{2\log y}\Big).$$

(Here we used the bound $\sqrt{2\pi\phi_2(c,y)} \ll \log x$ from Theorem 4.2 and $y > (\log x)^{1+\epsilon}$.) Since $y \le \exp((\log x)^{\frac{1}{4}})$, this yields the bound

$$\ll \frac{q_0^{-c+\epsilon}\Psi(x,y)}{(\log y)^3(1+|\gamma|x)^2}, \tag{6.10}$$

with plenty to spare.

Finally we consider the contribution of the vertical line segment given in (i). We split this integral into the regions $|t| \le 1/\log y$ and $1/\log y \le |t| \le y$. We first treat the saddle-point region $|t| \le 1/\log y$ lying near the real axis, which contributes to the main term of the formula in (2). Certainly $c+it = c+O(|t|)$, and we may check easily that for $|t| \le 1$

$$|H(c+it;q_0) - H(c;q_0)| \ll |t| q_0^{-c+\epsilon}.$$

It is clear that

$$\check{\Phi}(c+it,\gamma x) - \check{\Phi}(c,\gamma x) = \int_0^\infty \Phi(w) e(\gamma x w)(w^{c-1+it} - w^{c-1}) dw \ll |t|,$$

and integrating by parts twice we also have

$$\check{\Phi}(c+it,\gamma x) - \check{\Phi}(c,\gamma x) = \int_0^\infty \frac{d^2}{dw^2}\Big(\Phi(w) w^{c-1}(w^{it}-1)\Big)\frac{e(\gamma x w)}{(2\pi i x\gamma)^2} dw \ll \frac{|t|}{(|\gamma| x)^2}.$$

We conclude that $|\check{\Phi}(c+it,\gamma x) - \check{\Phi}(c,\gamma x)| \ll |t|/(1+|\gamma|x)^2$. Putting these observations together we see that for $|t| \le 1/\log y$,

$$|(c+it)\check{\Phi}(c+it,\gamma x)H(c+it;q_0) - c\check{\Phi}(c,\gamma x)H(c;q_0)| \ll \frac{q_0^{-c+\epsilon}}{(\log y)(1+|\gamma|x)^2}.$$

Hence the contribution of the region $|t| \le 1/\log y$ to $M(x,y;q_0,y)$ is

$$\frac{1}{2\pi i}\int_{c-i/\log y}^{c+i/\log y} \zeta(s;y)\frac{x^s}{s}\Big(cH(c;q_0)\check{\Phi}(c,\gamma x) + O_\epsilon\Big(\frac{q_0^{-c+\epsilon}}{(\log y)(1+|\gamma|x)^2}\Big)\Big) ds$$

$$= cH(c;q_0)\check{\Phi}(c,\gamma x)\Psi(x,y) + O_\epsilon\Big(\frac{q_0^{-c+\epsilon}}{(\log y)(1+|\gamma|x)^2}\Psi(x,y)\Big), \tag{6.11}$$

upon using Lemma 4.5 to produce the main term and the absolute value integral in Lemma 4.6 to bound the error term.

Next consider the remaining region $1/\log y \le |t| \le y$ in segment (i). Bounding the absolute value of the integrand, and using Lemma 4.4 and that $(1-c) \ll (\log\log x)/\log y$ (see (4.9)), this contribution is

$$\begin{aligned}
&\ll x^c q_0^{-c+\epsilon}\frac{y^3}{(1+|\gamma|x)^2} \max_{1/\log y \le |t| \le y} |\zeta(c+it;y)| \\
&\ll q_0^{-c+\epsilon}\frac{y^3}{(1+|\gamma|x)^2} x^c \zeta(c;y) \exp\Big(-C\frac{\log x}{(\log y)(\log\log x)^2}\Big),
\end{aligned}$$

for some positive constant C. Appealing now to Theorems 4.1 and 4.2, and using $y \le \exp((\log x)^{\frac{1}{2}-\delta})$, we deduce that the above is bounded by

$$\ll q_0^{-c+\epsilon}\Psi(x,y)\frac{y^3\log x}{(1+|\gamma|x)^2}\exp\Big(-C\frac{\log x}{(\log y)(\log\log x)^2}\Big) \le \frac{q_0^{-c+\epsilon}\Psi(x,y)}{(\log y)^3(1+|\gamma|x)^2}.$$

Combining this bound with (6.6), (6.7), (6.8), (6.10), (6.11), we obtain the estimate of the Proposition in the case when $|\gamma| \le x^{\delta-1}$.

$\square$

Proof of Theorem 2.4. We use Proposition 6.1 and Lemma 5.1, writing $q = q_0 q_1$ to obtain

$$E(x,y;\alpha) = \frac{\mu(q_1)}{\phi(q_1)} M(x,y;q_0,\gamma) + O\Big(x^{\frac{3}{4}+\epsilon}\Big). \tag{6.12}$$

Now Proposition 6.3 (1) applied to $M(x,y,q_0,\gamma)$ gives the bound of part (1). Next we note that the Hildebrand-Tenenbaum saddle point $c = c(x,y)$ satisfies $c = 1 - 1/\kappa + O(\log\log y/\log y)$ (see (4.11)), and so for $c_0 = 1 - \frac{1}{\kappa}$ we have

$$\begin{aligned}\frac{1}{q_0^c}\prod_{p|q_0}\Big(1 - \frac{p^c-1}{p-1}\Big)(c\check{\Phi}(c,\gamma x)) \quad &= \quad \frac{1}{q_0^{c_0}}\prod_{p|q_0}\Big(1 - \frac{p^{c_0}-1}{p-1}\Big)(c_0\check{\Phi}(c_0,\gamma x)) \\ &\quad + O_\epsilon\Big(\frac{q_0^{-c_0+\epsilon}}{(1+|\gamma|x)^2}\frac{\log\log y}{\log y}\Big).\end{aligned}$$

Now part (2) follows upon using this formula in the expression for $M(x,y,q_0,\gamma)$ in Proposition 6.3 (2), substituting the result into (6.12), noting that $\phi(q_1) \gg (q_1)^{-1+\epsilon}$. $\square$

7. Counting Weighted Smooth Solutions: Proof of Theorem 2.1

We initially suppose that $(\log x)^{2+\delta} \le y \le \exp((\log x)^{\frac{1}{2}-\delta})$, and we shall raise the lower bound on y as the proof progresses. We employ the Hardy-Littlewood circle method to evaluate

$$N(x,y;\Phi) = \int_0^1 E(x,y;\alpha)^2 E(x,y;-\alpha)d\alpha.$$

Let a fixed small number $\delta > 0$ be given, which we use as a parameter in defining major and minor arcs. Given a rational number a/q with $(a,q) = 1$ and $q \le x^{\frac{1}{4}}$, we define the major arc centered at a/q to be the set of all points $\alpha \in [0,1]$ with $|\alpha - a/q| \le x^{\delta-1}$. Note that any $\alpha \in [0,1]$ lies on at most one major arc. We will find it convenient to group the major arcs $[0, x^{\delta-1}]$ and $[1-x^{\delta-1}, 1]$ together, and on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ we may identify them with $[-x^{\delta-1}, x^{\delta-1}]$. The union of the major arcs is denoted $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ are defined to be the complement of the major arcs $[0,1]\backslash\mathfrak{M}$.

Suppose α lies on a minor arc. By Dirichlet's theorem on Diophantine approximation we may write $\alpha = a/q + \gamma$ where $q \le \sqrt{x}$, $(a,q) = 1$ and $|\gamma| \le 1/(q\sqrt{x})$. Since $\alpha \in \mathfrak{m}$ we must have that either $q > x^{\frac{1}{4}}$, or that $|\gamma| \ge x^{\delta-1}$. If the latter case holds then, using Propositions 6.1 and 6.3, we find that $E(x,y;\alpha) \ll x^{\frac{3}{4}+\epsilon}$. In the former case, Proposition 6.3 with (6.5) gives that

$$M(x,y;q,\gamma) \ll \Psi(x,y) q_0^{-c+\epsilon} q_1^{-1+\epsilon} \ll x^{\frac{3}{4}+\epsilon}.$$

Then by Proposition 6.1 we have $E(x,y;\alpha) \ll x^{\frac{3}{4}+\epsilon}$. Thus $E(x,y;\alpha) \ll x^{\frac{3}{4}+\epsilon}$ when α lies on a minor arc. Therefore

$$\int_{\mathfrak{m}} E(x,y;\alpha)^2 E(x,y;-\alpha)d\alpha \ll x^{\frac{3}{4}+\epsilon} \int_{\mathfrak{m}} |E(x,y;\alpha)|^2 d\alpha \ll x^{\frac{3}{4}+\epsilon} \int_0^1 |E(x,y;\alpha)|^2 d\alpha,$$

By Parseval, we have

$$\int_0^1 |E(x,y;\alpha)|^2 d\alpha = \sum_{n\in\mathcal{S}(y)} \left|\Phi\left(\frac{n}{x}\right)\right|^2 \ll E(x,y;0) \ll \Psi(x,y),$$

where the last inequality follows from Theorem 2.4, or alternatively from an application of Theorem 4.3. From this we obtain the minor arc bound

$$\int_{\mathfrak{m}} E(x,y;\alpha)^2 E(x,y;-\alpha)d\alpha \ll x^{\frac{3}{4}+\epsilon}\Psi(x,y). \tag{7.1}$$

It remains now to evaluate the major arc contribution. If $z = z_1 + O(z_2)$ then it follows that $|z|^2 z = |z_1|^2 z_1 + O(|z_2||z|^2)$. Therefore if α lies on the major arc centered at a/q, Proposition 6.1 gives that, with $\alpha = a/q + \gamma$ as before,

$$\begin{aligned} E(x,y;\alpha)^2 E(x,y;-\alpha) &= |E(x,y;\alpha)|^2 E(x,y;\alpha) \\ &= |M(x,y;q,\gamma)|^2 M(x,y;q,\gamma) + O_\epsilon(x^{3/4+\epsilon}|E(x,y;\alpha)|^2). \end{aligned}$$

Thus the major arc contribution is

$$\sum_{q\le x^{\frac{1}{4}}} \sum_{\substack{a=0\\(a,q)=1}}^{q-1} \int_{-x^{\delta-1}}^{x^{\delta-1}} |M(x,y;q,\gamma)|^2 M(x,y;q,\gamma)d\gamma + O_\epsilon\Big(x^{\frac{3}{4}+\epsilon}\int_0^1 |E(x,y;\alpha)|^2 d\alpha\Big),$$

which we may simplify to

$$\sum_{q\le x^{\frac{1}{4}}} \phi(q) \int_{-x^{\delta-1}}^{x^{\delta-1}} |M(x,y;q,\gamma)|^2 M(x,y;q,\gamma)d\gamma + O_\epsilon(x^{\frac{3}{4}+\epsilon}\Psi(x,y)). \tag{7.2}$$

Using the decomposition $q = q_0 q_1$ of Lemma 6.2 and the estimate of Proposition 6.3 (2), we find that $|M(x,y;q,\gamma)|^2 M(x,y;q,\gamma)$ equals

$$\begin{aligned} &\frac{\mu(q_1)}{\phi(q_1)^3}\frac{1}{q_0^{3c}}\prod_{p|q_0}\Big(1-\frac{p^c-1}{p-1}\Big)^3 c^3|\check{\Phi}(c,\gamma x)|^2\check{\Phi}(c,\gamma x)\Psi(x,y)^3 \\ &+O_\epsilon\Big(\frac{1}{\phi(q_1)^3}\sum_{j=1}^{3}\Big(x^{\frac{3}{4}+\epsilon}q_0^{-\frac{3}{4}+\epsilon}+\frac{\Psi(x,y)q_0^{-c+\epsilon}}{(\log y)(1+|\gamma|x)^2}\Big)^j\Big(\frac{1}{q_0^c}\Psi(x,y)\check{\Phi}(c,\gamma x)\Big)^{3-j}\Big). \end{aligned}$$

Since $\sum_{j=1}^3 A^j B^{3-j} \ll A^3 + AB^2$, and since $|\check{\Phi}(c,\gamma x)| = O(1)$ (using Lemma 3.3) we may simplify the error term above to

$$\ll \frac{1}{\phi(q_1)^3}\Big(x^{\frac{9}{4}+3\epsilon}q_0^{-\frac{9}{4}+3\epsilon} + x^{\frac{3}{4}+\epsilon}q_0^{-2c-\frac{3}{4}+\epsilon}\Psi(x,y)^2 + \frac{\Psi(x,y)^3 q_0^{-3c+\epsilon}}{(\log y)(1+|\gamma|x)^2}\Big).$$

To upper bound the contribution of this error term to (7.2), we note that

$$\int_{-x^{\delta-1}}^{x^{\delta-1}} \frac{d\gamma}{(1+|\gamma x|)^2} \ll \frac{1}{x},$$

and then we obtain
(7.3)

$$\ll \sum_{q \le x^{\frac{1}{4}}} \frac{\phi(q)}{\phi(q_1)^3}\Big(x^{\frac{5}{4}+\delta+\epsilon} q_0^{-\frac{9}{4}+\epsilon} + x^{-\frac{1}{4}+\delta+\epsilon} q_0^{-2c-\frac{3}{4}+\epsilon}\Psi(x,y)^2 + \frac{\Psi(x,y)^3 q_0^{-3c+\epsilon}}{x\log y}\Big).$$

We now raise the lower bound to $y \ge (\log x)^{4+8\delta}$. We then have $\Psi(x,y) \ge x^{\frac{3}{4}+\frac{3}{2}\delta}$, and in addition the Hildebrand-Tenenbaum saddle point $c > \frac{3}{4}$ by (4.11). We deduce for y in this range the error term contribution (7.3) above is

$$\ll \frac{\Psi(x,y)^3}{x\log y}.$$

We conclude that for $y \ge (\log x)^{4+8\delta}$ the major arcs contribution is

$$\Psi(x,y)^3 \sum_{q \le x^{\frac{1}{4}}} \frac{\mu(q_1)}{\phi(q_1)^2}\frac{\phi(q_0)}{q_0^{3c}} \prod_{p | q_0}\Big(1 - \frac{p^c-1}{p-1}\Big)^3 \int_{-x^{\delta-1}}^{x^{\delta-1}} c^3 |\check{\Phi}(c,\gamma x)|^2 \check{\Phi}(c,\gamma x) d\gamma$$
$$+ O_\epsilon\Big(x^{\frac{3}{4}+\epsilon}\Psi(x,y) + \frac{\Psi(x,y)^3}{x\log y}\Big).$$

Using Lemma 3.3 with $k = 2$, and the Plancherel formula, we obtain that

$$\begin{aligned}
\int_{-x^{\delta-1}}^{x^{\delta-1}} c^3|\check{\Phi}(c,\gamma x)|^2 \check{\Phi}(c,\gamma x)d\gamma &= \frac{c^3}{x}\int_{-x^\delta}^{x^\delta} |\check{\Phi}(c,\xi)|^2 \check{\Phi}(c,\xi) d\xi \\
&= \frac{c^3}{x}\Big(\int_{-\infty}^{\infty} |\check{\Phi}(c,\xi)|^2 \check{\Phi}(c,\xi) d\xi + O\Big(\int_{|\xi| > x^\delta} \frac{1}{1+\xi^2} d\xi\Big)\Big) \\
&= \frac{c^3}{x}\Big(\int_0^\infty \int_0^\infty \Phi(t_1)\Phi(t_2)\Phi(t_1+t_2)(t_1 t_2(t_1+t_2))^{c-1} dt_1 dt_2 + O(x^{-\delta})\Big) \\
&= \frac{\mathfrak{S}_\infty(c,\Phi)}{x} + O(x^{-1-\delta}).
\end{aligned}$$

For $y \geq (\log x)^{4+8\delta}$, using $c \geq 3/4$ we see that

$$\begin{aligned}\sum_{q \leq x^{\frac{1}{4}}} \frac{\mu(q_1)}{\phi(q_1)^2}\frac{\phi(q_0)}{q_0^{3c}} & \prod_{p|q_0}\Big(1-\frac{p^c-1}{p-1}\Big)^3 \\ &= \sum_{q=1}^{\infty} \frac{\mu(q_1)}{\phi(q_1)^2}\frac{\phi(q_0)}{q_0^{3c}} \prod_{p|q_0}\Big(1-\frac{p^c-1}{p-1}\Big)^3 + O\Big(\sum_{q>x^{\frac{1}{4}}} \frac{1}{q^{3c-1-\epsilon}}\Big) \\ &= \Big(\sum_{q_0 \in \mathcal{S}(y)} \frac{\phi(q_0)}{q_0^{3c}} \prod_{p|q_0}\Big(1-\frac{p^c-1}{p-1}\Big)^3\Big)\Big(\sum_{\substack{q_1 \\ p|q_1 \Rightarrow p>y}} \frac{\mu(q_1)}{\phi(q_1)^2}\Big) + O(x^{-\frac{1}{16}}) \\ &= \prod_{p \leq y}\Big(1+\frac{p-1}{p(p^{3c-1}-1)}\Big(\frac{p-p^c}{p-1}\Big)^3\Big) \prod_{p>y}\Big(1-\frac{1}{(p-1)^2}\Big) + O(x^{-\frac{1}{16}}) \\ &= \mathfrak{S}_f(c,y) + O(x^{-\frac{1}{16}}).\end{aligned}$$

Putting these remarks together, we conclude that for $y \geq (\log x)^{4+8\delta}$ the major arcs contribution is

$$\mathfrak{S}_\infty(c,\Phi)\mathfrak{S}_f(c,y)\frac{\Psi(x,y)^3}{x} + O_\epsilon\Big(x^{\frac{3}{4}+\epsilon}\Psi(x,y) + \frac{\Psi(x,y)^3}{x\log y}\Big).$$

We combine this result with the minor arcs estimate (7.1) to conclude that (7.4)

$$N(x,y;\Phi) = \mathfrak{S}_\infty(c,\Phi)\mathfrak{S}_f(c,y)\frac{\Psi(x,y)^3}{x} + O_\epsilon\Big(x^{\frac{3}{4}+\epsilon}\Psi(x,y) + \frac{\Psi(x,y)^3}{x\log y}\Big).$$

To obtain an asymptotic formula, we now impose the lower bound $y \geq (\log x)^{8+\delta}$. Thus $\kappa \geq 8+\delta$, so that $\Psi(x,y) = x^{1-1/\kappa+o(1)} > x^{\frac{7}{8}+\epsilon}$. Now by (4.11) we know that $c = 1-1/\kappa + O(\log\log y/\log y)$. Both $\mathfrak{S}_f(c,y)$ and $\mathfrak{S}_\infty(c,\Phi)$ are of constant size, and moreover we have

$$\mathfrak{S}_f(c,y) = \mathfrak{S}\Big(1-\frac{1}{\kappa},y\Big) + O\Big(\frac{\log\log y}{\log y}\Big),$$

and

$$\mathfrak{S}_\infty(c,\Phi) = \mathfrak{S}_\infty\Big(1-\frac{1}{\kappa},\Phi\Big) + O\Big(\frac{\log\log y}{\log y}\Big).$$

We use these observations in (7.4), and note also that the lower bound on $\Psi(x,y)$ above implies that the error term $x^{\frac{3}{4}+\epsilon}\Psi(x,y)$ is subordinate to the error term $\Psi(x,y)^3/(x\log y)$, so that (7.4) is an asymptotic formula. This proves Theorem 2.1.

8. Counting Weighted Primitive Smooth Solutions: Proof of Theorem 2.2

We suppose $(\log x)^{8+\delta} \leq y \leq \exp\Big((\log x)^{\frac{1}{2}-\delta}\Big)$. Let $z = \frac{1}{2}\log y$, and put $P_z = \prod_{p \leq z} p$. By the prime number theorem we know that $P_z = e^{z+o(z)} \leq y$.

We assert that

$$\Big|N^*(x,y;\Phi) - \sum_{d|P_z} \mu(d) N\Big(\frac{x}{d}, y; \Phi\Big)\Big| \le \sum_{z<p\le y} N\Big(\frac{x}{p}, y; |\Phi|\Big). \tag{8.1}$$

To establish (8.1), it suffices to observe that its left hand side counts weighted solutions to $X+Y=Z$ with $XYZ \in \mathcal{S}(y)$ and such that the gcd (X,Y,Z) is an integer greater than 1 and divisible only by primes larger than z. The proof will derive the desired asymptotic formula for the inclusion-exclusion sum $\sum_{d|P_z} \mu(d) N\Big(x/d, y; \Phi\Big)$ on the left side of (8.1) and will complete the argument by showing that the right side of (8.1) is small compared to this asymptotic estimate.

To handle the terms arising in (8.1) we first consider $N(x/k, y; \Phi)$ and $N(x/k, y; |\Phi|)$ where $1 \le k \le y$. In our range for x and y, the exponent $\kappa = \kappa(x,y) := \log y / \log\log x$ satisfies

$$\Big|\frac{1}{\kappa(x,y)} - \frac{1}{\kappa(x/k,y)}\Big| \le \frac{\log\log x - \log\log(x/y)}{\log y} \ll \frac{1}{\log x},$$

and therefore

$$\begin{aligned}\mathfrak{S}_\infty\Big(1 - \frac{\log\log(x/k}{\log y}, \Phi\Big)\mathfrak{S}_f\Big(1 - \frac{\log\log(x/k)}{\log y}, y\Big)&\\ = \mathfrak{S}_\infty\Big(1 - \frac{\log\log x}{\log y}, \Phi\Big)\mathfrak{S}_f\Big(1 - \frac{\log\log x}{\log y}, y\Big) &+ O\Big(\frac{1}{\log x}\Big).\end{aligned}$$

Furthermore, by Theorem 4.3 we have that

$$\Psi(\frac{x}{k}, y) = k^{-c(x/k,y)} \Psi(x,y)\Big(1 + O\Big(\frac{\log y}{\log x}\Big)\Big).,$$

where $c(x/k, y)$ is the Hildebrand-Tenenbaum saddle point. Using Theorem 2.1 we conclude that

$$\begin{aligned} N\Big(\frac{x}{k}, y; \Phi\Big) = \mathfrak{S}_\infty\Big(1 - \frac{\log\log x}{\log y}, \Phi\Big)\mathfrak{S}_f\Big(1 - \frac{\log\log x}{\log y}, y\Big)\frac{\Psi(x,y)^3}{x} k^{1-3c(x/k,y)}&\\ + O\Big(k^{1-3c(x/k,y)} \frac{\Psi(x,y)^3 \log\log y}{x \log y}\Big).& \end{aligned} \tag{8.2}$$

Similarly, we obtain the upper bound

$$N(\frac{x}{k}, y; |\Phi|) \ll k^{1-3c(x/k,y)} \frac{\Psi(x,y)^3}{x}. \tag{8.3}$$

We first bound the right hand side of (8.1). We find using (8.3) that it is bounded by

$$\ll \frac{\Psi(x,y)^3}{x} \sum_{z<p\le y} p^{1-3c(x/p,y)},$$

Since by (4.11) we have

$c(x/p, y) = 1 - 1/\kappa(x/p, y) + O(\log\log y / \log y) = 1 - 1/\kappa + O(\log\log y/\log y) > 3/4,$

the above is bounded by

$$\ll \frac{\Psi(x,y)^3}{x} \sum_{z<p\le y} p^{-5/4} \ll \frac{\Psi(x,y)^3}{xz^{\frac14}}.$$

Now, using (8.2), we treat the sum on the left side in (8.1), and find that

$$\sum_{d|P_z} \mu(d) N\Big(\frac{x}{d}, y; \Phi\Big) = \mathfrak{S}_\infty\Big(1-\frac{1}{\kappa}, \Phi\Big)\mathfrak{S}_f\Big(1-\frac{1}{\kappa}, y\Big)\frac{\Psi(x,y)^3}{x}\Big(\sum_{d|P_z}\mu(d) d^{1-3c(x/d,y)}\Big)$$
$$+ O\Big(\frac{\Psi(x,y)^3 \log\log y}{x\log y}\Big(\sum_{d|P_z} d^{1-3c(x/d,y)}\Big)\Big).$$

Since $c(x/d,y) > 3/4$, as noted above, the remainder term here is $O(\Psi(x,y)^3 \log\log y/(x\log y))$. Next we treat the sum appearing in this last estimate, and again we use that $c(x/d,y) = 1-1/\kappa + O(\log\log y/\log y)$. We obtain

$$\begin{aligned}\sum_{d|P_z}\mu(d)d^{1-3c(x/d,y)} &= \sum_{d\le z}\mu(d)d^{3/\kappa-2}\Big(1+O\Big(\frac{\log d\log\log y}{\log y}\Big)\Big) + O\Big(\sum_{d>z} d^{3/\kappa-2+o(1)}\Big)\\ &= \prod_{p\le z}\Big(1-\frac{1}{p^{2-3/\kappa}}\Big) + O\Big(\frac{\log\log y}{\log y}\Big) + O(z^{-1+3/\kappa+o(1)})\\ &= \prod_{p\le y}\Big(1-\frac{1}{p^{2-3/\kappa}}\Big) + O((\log y)^{-\frac12}).\end{aligned}$$

We now define

$$\begin{aligned}\mathfrak{S}_f^*(c,y) &:= \mathfrak{S}_f(c,y)\prod_{p\le y}\Big(1-\frac{1}{p^{3c-1}}\Big)\\ &= \prod_{p\le y}\Big(1+\frac{1}{p^{3c-1}}\Big(\frac{p-1}{p}\Big(\frac{p-p^c}{p-1}\Big)^3-1\Big)\prod_{p>y}\Big(1-\frac{1}{(p-1)^2}\Big).\end{aligned} \tag{8.4}$$

Using this definition, substition of the above estimates in (8.1) gives

$$N^*(x,y;\Phi) = \mathfrak{S}_\infty\Big(1-\frac{1}{\kappa},\Phi\Big)\mathfrak{S}_f^*\Big(1-\frac{1}{\kappa},y\Big)\frac{\Psi(x,y)^3}{x} + O\Big(\frac{\Psi(x,y)^3}{x(\log y)^{\frac14}}\Big).$$

This proves Theorem 2.2.

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